Oklahoma Mathematical Inquiry Project
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A Teacher's Guide on How to Promote Perseverance and Productive Problem-Solving Skills in the Mathematics Classroom

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Many students struggle with negative attitudes toward math, leading to disengagement and lower achievement. This paper explores how to transform this experience by examining the impact of emotions on problem-solving through affective pathways. It introduces the MIP Guiding Principles as a framework for cultivating a supportive classroom environment and provides actionable strategies that leverage these principles. By designing engaging activities, teachers challenge students to persevere and develop effective problem-solving skills. Furthermore, by establishing sociomathematical norms, teachers foster a collaborative mathematical culture. Through these approaches, students are likely to become more engaged, develop a positive math identity, and ultimately achieve greater success.

Introduction

Teaching mathematics is both an art and a science. Teachers are tasked with helping their students understand mathematical facts and concepts, real-world applicability, critical thinking skills, and reasoning. Teaching these skills requires special attention to a myriad of details, such as the sequencing of learning objectives—all of which build upon other prerequisite skills and necessitate time and practice to master. While teachers must certainly attend to their students' *cognition*, other unseen *affective* forces such as emotions, moods, and feelings are simultaneously at play, either enhancing or undermining student motivation. Evidence suggests that *all* learning is a complex activity involving connections between the brain's cognitive and emotional centers (Damasio, 2006). As such, another—often unacknowledged—role of the mathematics teacher is helping students successfully navigate the many emotional states experienced in the learning process *en route* to a reasonable solution.

The affective pathways in problem solving

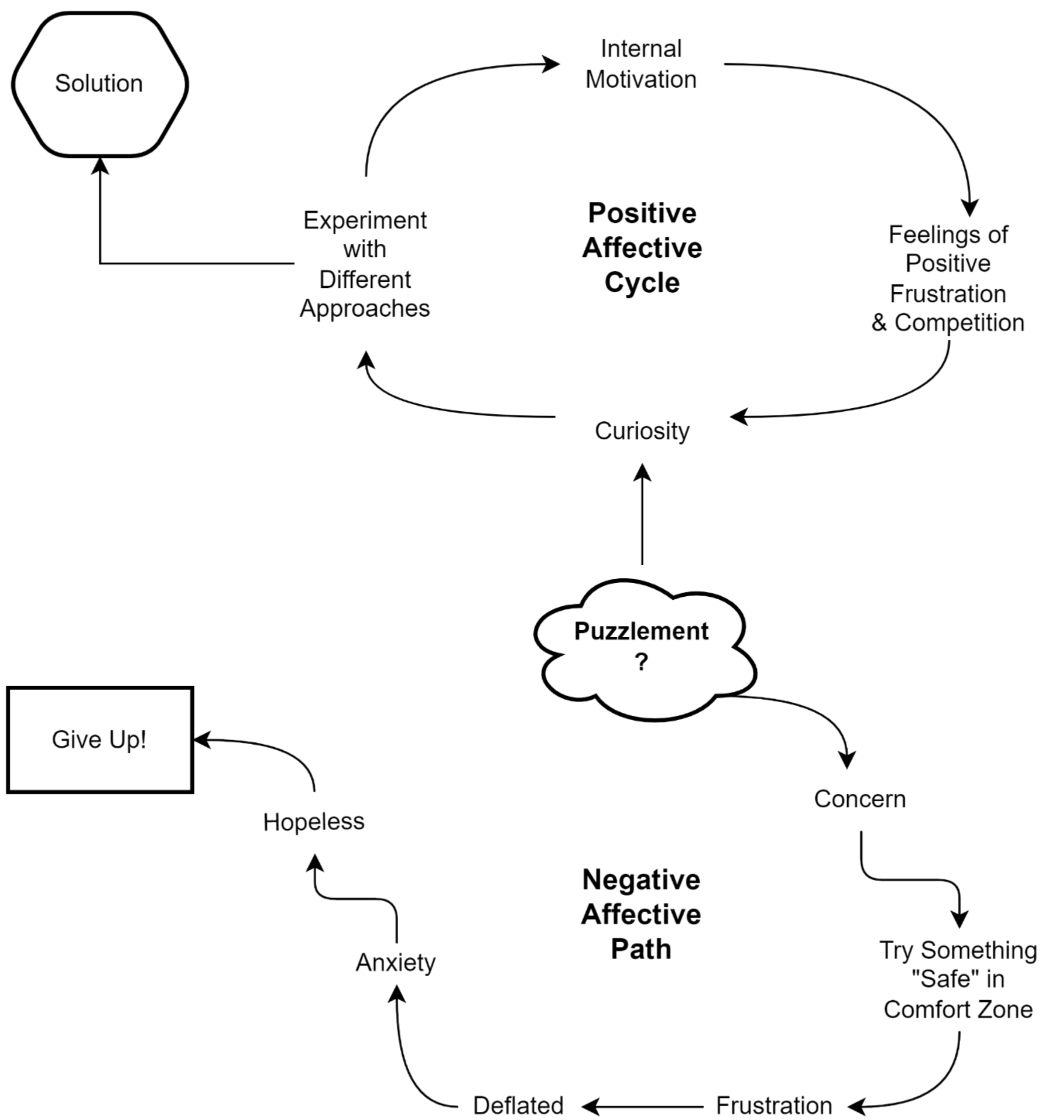
Underlying feelings, emotions, or moods that students experience influence their choices in both the micro and the macro: a student's emotional state affects their decisions on how to approach a specific math problem as well as their decisions on whether or not to pursue further study in mathematics. Since problem solving is, in essence, a series of decisions, the emotions and affective constructs that accompany the problem-solving process critically influence a student's learning and are thus worthy of special attention.

Students who are successful in mathematics will generally demonstrate two essential affective qualities: a positive view of *productive struggle* and the ability to *persevere* through complex tasks. We will look at these two traits shortly, but first, we describe two emotional journeys that begin in the same place but conclude with vastly different outcomes.

DeBellis and Goldin (2006) describe a **positive affective cycle** where the student responds to the challenge with curiosity, puzzlement, and positive feelings that create internal motivation to understand the problem more thoroughly. As the student tries different approaches, they may feel frustrated, nudged forward by healthy self-competition, strategic rethinking, and satisfaction as they gain a better understanding of the problem via new approaches. Despite a myriad of different emotional states, the student *perseveres*! This emotional pathway can repeatedly cycle until, at last, the student is convinced they have a solution. At this point, the student experiences positive feelings of relief and accomplishment that boost their confidence and bolster their identity as a mathematical problem solver.

On the flip side, students may experience a **negative affective path** that likewise begins with curiosity and puzzlement but fails to produce the desired learning. In this scenario, the student may 'play it safe' and only explore a procedure with which they are comfortable. If they fail in their 'safe' attempt, they may experience frustration, followed by anxiety and hopelessness. Once feelings of hopelessness surface, the student gives up. (See *Figure 1*.)

Figure 1. A positive affective cycle versus a negative affective path



Mathematics teachers play significant roles in helping students navigate these two emotional pathways. One path leads to successful problem solving and hope for future success; the other leads to disappointment and disidentification with mathematics, which can lead to the belief that "I'm just not a math person."

Rebranding mathematics by embracing the MIP guiding principles

Throughout multiple years of schooling, students tend to develop false beliefs about what mathematics is all about. For example, the bite-sized nature of most mathematics curricula gives students the impression that mathematical methods and answers are not to be figured out themselves but provided by 'experts' on high (Carpenter et al., 1983; National Assessment of Educational Progress, 1983). Students' limited experience with authentic problem solving downgrades their notion of mathematics to the memorization and application of formulas that they do not necessarily understand.

Mathematicians know that mathematics is far more than memorizing formulas and *plug-and-chug*. Genuine problem solving is an active and creative process that takes time. Teachers can work to change students' perceptions of mathematics and what successful problem solving requires of them by embracing the three guiding principles of the Mathematical Inquiry Project (MIP): (1) Active Learning, (2) Meaningful Applications, and (3) Academic Success Skills (The Mathematical Inquiry Project, 2023). The rebranding of math and mathematical culture changes students' expectations and promotes *perseverance* and positive *affect* during *productive struggle*. In the following sections, we discuss the role of each MIP principle in light of its impact on student learning, particularly on promoting positive affect.

Active Learning

> *"Students learn through engaging in deep problems requiring them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned."*

Active learning involves students directly participating in the learning process. Students are more engaged and involved in understanding the material when actively solving problems that align with the core concepts. As students select, perform, and evaluate actions linked to the concepts they are learning, they develop a deep understanding of the material. Regular exposure to such activities enhances confidence and fortifies students against anxiety typically associated with the initial 'puzzlement' stages of mathematical problem solving, illustrated in *Figure 1*.

Active learning combats boredom and disengagement by fostering a positive affective cycle. Students grapple with the material, experiencing positive frustration that fuels progress toward solutions and deeper understanding. Positive reinforcement through feedback and recognition strengthens their self-efficacy and motivation, leading to a desire for continued participation. This cycle is further bolstered by active learning's ability to cultivate transferable knowledge (applying concepts to new contexts) and a safe space for exploration, prediction, and reflection. Students experiment with different approaches, select the appropriate tools to solve problems, and refine their understanding, thereby mirroring the positive affective cycle. This fosters long-term retention and intrinsic motivation.

Active learning also directly addresses the misconception that math is about memorizing arbitrary sequences. By engaging students in experiences that require them to "select, perform, and evaluate actions whose structures are equivalent to the concepts to be learned," this approach dismantles that myth. Selecting the right approach goes beyond just choosing a formula. It involves selecting relevant information within a problem, similar to choosing the appropriate building materials for a project. This emphasizes that there's no "one-size-fits-all" approach in math, mirroring the experimentation within the positive affective cycle. Evaluating their actions and comparing approaches with peers dismantles the idea of blind memorization. Students see the underlying structure and relationships between concepts, realizing that chosen actions (like following a formula) are not arbitrary but correspond to the mathematical concept they're solving. This aligns with the positive affective cycle's emphasis on reflection, where students refine their understanding by reevaluating their approaches. In essence, by actively selecting information, performing actions, and evaluating their effectiveness, students move away from rote memorization and towards a deeper comprehension of math. They become active participants, recognizing the underlying logic that ties different approaches together. This fosters a positive learning experience and combats the misconception that math is a collection of arbitrary steps.

For types of problems that allow for different approaches, see Example B and Example E.

Meaningful Applications

> *"Applications are incorporated in mathematics classes to support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure."*

Integrating *meaningful applications* into mathematics curriculum highlights the relevance of mathematical concepts and actively engages students' intuitive reasoning. It allows them to discern the pervasive role of math across various fields and situations, imparting a more profound sense of relatability and significance. These contexts act as catalysts for concrete reasoning, prompting students to build mental frameworks that later translate into abstract representations via variables, expressions, diagrams, and graphs.

Meaningful applications are fertile ground for fostering critical thinking skills. Equations and models derived from practical tasks embody meaningful relationships ripe for exploration, justification, and questioning. These contextual scenarios prompt concrete reasoning, enabling students to engage in mental constructions that can subsequently be abstracted into higher-order reasoning skills. Understanding how math applies in diverse scenarios enriches comprehension and cultivates a positive perception of its practical utility and relevance. As a result, students exhibit more buy-in, motivation, and positive affect towards learning the subject. The inherent usefulness of the problem also encourages the student to persevere to the end.

An example of an activity that exemplifies the *meaningful application* component is one that allows students to use a spreadsheet for computations. By utilizing tools like Excel or Google Sheets, students can minimize the cognitive load from manual computations and focus on how changes in the input affect the output. This approach eliminates tedious computations, enabling students to perform "what if" analyses and explore multiple scenarios to justify their claims much quicker. Consequently, they are more likely to persist and discover mathematical relationships between the input and output. Providing students with these tools helps them concentrate on the big picture, which may increase their confidence and keep them in a positive affective cycle. Refer to Example F and this link for specific activities that utilize spreadsheets.

Academic Success Skills

> *"Students construct an identity as learners in ways that enable productive engagement in their education and the associated academic community."*

Helping students construct a positive identity as learners in the mathematics classroom is crucial. When they feel a sense of belonging and competence, they are more likely to engage actively and persistently in their learning. Likewise, being part of an academic community encourages collaboration, support, and shared learning experiences. Feeling connected and supported by peers and educators creates an environment that embraces the productive struggle essential for learning math.

In addition to fostering a supportive community, the competencies and skill sets that students credit for their successes—as well as the deficiencies they blame for their failures—directly impact their approach to learning mathematics. Attribution Theory, which examines how individuals explain their successes and failures, plays a critical role in *academic success skills*. It significantly influences how students develop their identities as learners, a core aspect of the MIP framework. When students attribute their achievements in math to effort, ability, and effective strategies (internal attributions), and view challenges as opportunities to deepen understanding (growth mindset), they cultivate a positive self-image as learners. This aligns with the MIP's emphasis on fostering a positive identity. Conversely, attributing failures to external factors like luck, task difficulty, or teacher ineffectiveness (external attributions) can lead to a negative self-perception in math. In a learning environment that highlights effort and provides specific feedback on progress and strategies, students develop a growth mindset and engage in collaboration. This positive cycle fosters social attributions of success, empowering students to take ownership of their learning journey and develop a strong identity as capable math learners.

Tying it All Together

Faculty are encouraged to incorporate the MIP guiding principles into their classrooms whenever possible. They may do so by utilizing activities that are published on the Oklahoma Mathematical Project website (okmip.com), which are specifically designed with these principles in mind.

It's important to note that instructors do not need to completely redesign their courses to expose their students to the components of active learning, meaningful applications, and academic success skills. Below is an example of how one might begin adding content with these principles into their courses.

Old Method: Suppose the lesson for a Pre-Calculus course involves the Pythagorean Identities. The instructor may simply provide the following formulas and tell their students that they need to memorize them.

$$sin^2 x + cos^2 x = 1$$

$$1 + cot^2 x = csc^2 x$$

$$tan^2 x + 1 = sec^2 x$$

Notice that students are completely passive in this scenario.

New Method: Consider providing the first equation and encourage students to memorize it. Then, instead of memorizing the other equations, derive the second equation using the first equation. That is, show students that if each term in the first equation is divided by $\sin x$ and simplified, the second equation follows. Lastly, ask the students to derive the third equation using a similar method. Students should also be encouraged to share their work with their peers in groups.

In making this relatively simple change to how one presents material, students will become actively engaged with the content. For example, in this scenario, students will be actively learning via a meaningful application as they will need to *select* the correct trigonometric function to divide each term by, *perform* the operation of division, and simplify each fraction using their *prior knowledge* of the quotient identities, and *justify* their work to their peers. Allowing students to derive a new trigonometric identity may help them gain confidence in their ability to do math and help establish their *identities* as learners of mathematics.

Teacher actions for establishing classroom culture

The effective implementation of the three MIP components of inquiry relies on a teacher's capacity to cultivate a classroom culture that seamlessly embraces and supports these principles. This involves creating an environment that nurtures positive affect, productive struggle, and perseverance. This section will delineate essential teacher actions vital to this endeavor.

- Set realistic expectations regarding time: According to the literature, students learn to do mathematics best in environments where they are allowed ample time to grapple with and explain non-procedural tasks (NCTM, 2014; Schoenfeld, 1992).
  - Set the expectation that mathematical problem solving requires both creativity and time. This is done by choosing activities that cannot be solved immediately or with well-known methods.
  - Classroom culture must support the notion that *productive struggle* is acceptable and that tasks often take longer than the standard two- to three-minute homework problem.
- Set the expectation of multiple approaches:
  - Allow students to use more than one strategy to approach the task. (See Problem-solving strategies.) This flexibility will allow students to use their background, prior knowledge, and experience to tackle the task.
  - Allow for various mathematical tools and representations of the problem to enhance the students' understanding. Helpful digital tools include Excel, Desmos, GeoGebra, WolframAlpha, and Mathematica.
- Set the expectation of frustration: Teachers serve a vital role in nurturing students' perseverance in problem solving. Incorporating the motto, "If you are not struggling, you are not learning" (Carter, 2008, p. 136), establishes an environment that champions perseverance and underscores the necessity for struggle in learning.
  - Set the expectation that learning math takes practice and repeated effort and that feelings of frustration are a natural and necessary component in the learning process. People don't learn to play the piano after an hour at a keyboard; neither do they learn math after an hour at a whiteboard.

- Explicitly introduce the notion of *productive struggle* when presenting activities, emphasizing the expectation that overcoming challenges is a normal and valued part of the learning process. (See Sociomathematical Norms.)
- Develop clear and comprehensive classroom guidelines that delineate what constitutes acceptable mathematical justification (Yackel & Cobb, 1996). (See Sociomathematical Norms.)

Choosing Quality Learning Activities

To encourage our students to think critically, we need to choose quality tasks. According to Liljedahl (2021), "Good problem-solving tasks require students to get stuck and then to think, to experiment, to try and to fail, and to apply their knowledge in novel ways in order to get unstuck." Below are some characteristics of "good teacher tasks" that were compiled from CDE (1985), Cross et al. (2012), Kisker et al. (2012), Liljedahl (2021), Moschkovich (1999, 2011), NCTM (2014), and Schoenfeld (1992):

- Select tasks that encourage critical thinking. (See Meaningful Applications.)
- Choose learning activities that provide opportunities for authentic problem solving with multiple avenues for solutions. (See Example B, Example D, Example E, and Example F.)
- Use realistic data or real-world situations so that the task will have more meaning to the students. For example, consider creating tasks around a relatable storyline to better engage students. Constructing tasks your students can identify with will increase their interest and persistence. (See Example F.)
- Select activities that cannot be solved immediately or with well-known methods.
- Periodically introduce intentionally *ill-structured* learning activities. These types of activities best approximate the types of problem solving done by mathematicians. Exposing students to *ill-structured* activities forces them to be creative and to use any tools at their disposal. Evidence also suggests that it promotes a positive *affect* towards mathematics while supporting perseverance through productive problem solving (Kapur & Lee, 2009). (See Designing ill-structured learning activities with example.)

Designing ill-structured learning activities with example

According to theories of impasse-driven learning (VanLehn et al., 2003), mathematics instructors are often too quick to provide direct feedback when students make mistakes. The situation is like the parable of the caterpillar in the cocoon, which is prematurely 'helped' out of its silky case by a misguided little boy who feels sorry for it. In the end, the caterpillar never becomes a butterfly because the struggle in the cocoon was meant to be the agency of radical transformation of its body. In the same way, when teachers give too much 'help' too early—whether as feedback or direct instruction—students miss the productive struggle that their brains require to establish new neural learning pathways.

Kapur & Lee (2009) argue for 'a delay in structure' in learning and problem-solving situations and suggest that mathematics teachers employ *ill-structured* exercises.

An *ill-structured* exercise, by implication, is a challenging task because its solution is not prescribed (literally: 'written beforehand.') Being assigned an *ill-structured* activity might evoke conflicting or contradictory emotions in the student. Thus, *affect* is a powerful force that either helps or handicaps students when working on challenging mathematical problems. To help a student navigate the initial feelings of frustration, a teacher needs to frame the exercise by setting realistic expectations of what doing mathematics looks and feels like. (See Teacher Actions for Establishing Classroom Culture.) An example of an ill-defined problem is given in Example D.

Teacher actions that promote perseverance

Not only is the written task important, but also the implementation of that task. Both requiring our students to think and allowing them enough time to think is crucial. Liljedahl (2021) states, "Thinking is a necessary precursor of learning, and if students are not thinking, they are not learning."

Perseverance, like mathematics itself, can be developed in community. The following section outlines teacher actions for implementing successful small groups in the mathematics classroom.

In addition to allowing students sufficient time to think about and explore tasks, teachers can encourage students to work collaboratively. Compiling ideas from Hufford-Ackles et al. (2014), Lester et al. (1989), Liljedahl (2021), NCTM (2014), Schoenfeld (1992), and Smith (2000), consider the following when implementing tasks for group work:

- Manage Time Expectations:
    - Allow all students the chance to read the problem on their own and give them time to digest the nature of the problem. Once everyone has read the problem, the teacher or a student may also read it aloud.
    - Allow the students adequate time to work on the problem as a group. Give them space. That is, do not visit the individual groups until the students have had time to struggle with the problem.
    - Beware of the temptation to give help too early in the process.
- Manage language and cultural issues:
    - Answer any questions regarding the vocabulary used and the intent of the problem as needed. Be especially aware of cultural differences in the classroom. For example, if a related rates calculus problem involves a baseball diamond, international students may be unfamiliar with the layout of a baseball field or the rules by which the runners proceed around the bases. Be prepared to explain any cultural context required for understanding the task.
- Manage Group Dynamics:
    - Assign roles to each member of the group, such as:
        - A facilitator who keeps the group on task
        - A recorder who records the solutions for the team
        - A spokesperson who presents solutions to other groups or the instructor
    - Encourage the groups to discuss strategies to tackle the current task. If needed, turn this into a classroom discussion.
    - Require students to ask questions within their group first. Questions asked of the teacher must come from the group's spokesperson.
    - Ask each group questions about their work while circulating the room. Ask for specific details on what they are doing, why they are doing it, and how it will help them achieve their goal.

    - Offer hints and extensions as needed. If a group is frustrated where it is no longer productive, it may be beneficial to offer a more detailed and specific suggestion or strategy to get the group back on track.
- Enforce Sociomathematical Norms:
    - Ask the students to write up a complete solution to the problem. The answer should include all justifications, including verification that it 'makes sense.'
    - Provide each group a chance to explain their solution orally to the class if time allows. If a group is having trouble, have another group assist.
    - Praise the process, not just the correct answer.

Sociomathematical Norms

Sociomathematical norms refer to the established guidelines for how students and instructors should interact in a mathematics classroom. Unlike general classroom social norms, which apply universally, such as respect for peers and class participation, sociomathematical norms are uniquely focused on the development and justification of mathematical concepts and reasoning (Yackely & Cobb, 1996; Arnold & Norton, 2023). In this model, students learn to

- Engage in discussions where logical arguments are exchanged and tested to establish consensus.
- Think out loud even when mathematical ideas are in the developmental stage.
- Transform mathematical symbols and terminology into comprehensible concepts.
- Rely on logical reasoning and sense-making to assess argument validity.
- Adapt everyday language to align with mathematical logic and precision.

Establishing sociomathematical norms fosters a positive affect and cultivates greater perseverance, encouraging students to embrace productive struggle.

Problem-solving strategies

Creating tasks that encourage the application and expansion of prior knowledge while allowing flexibility in problem-solving strategies poses a challenge. When designing such tasks, it is essential to remember that the problem-solving process is more important than the solution itself. This section provides a list of problem-solving strategies students may implement while working on mathematical tasks. Many, if not all, of these strategies will be familiar to the reader, but it is helpful to have them compiled in one location for reference. The below strategies were assembled from works by Schoenfeld (1992) and Posamentier et al. (2015).

- Guessing-and-checking logically
- Working Backwards
- Recognizing Patterns
- Considering Extreme or Special Cases (See Example A.)
- Adopting a Different Point of View (See Example B and Example E.)
- Solving a Similar Analogous Problem (See Example C.)
- Organizing Data
- Making a Visual Representation

Illustrate examples of problem solving

As previously mentioned, the above strategies are not necessarily new to the reader, but a few examples may provide fresh insight into how they can be used. The examples provided below are adaptations of problems and solutions found in *Problem-Solving Strategies in Mathematics: From Common Approaches to Exemplary Strategies* by A. S. Posamentier and S. Krulik.

The following example (Example A) utilizes the strategy of considering extreme cases.

Example A: Jordan decides to see if a classmate could determine five whole numbers written down on paper. The goal for the classmate is to demonstrate their understanding of mean, median, and mode. The classmate is told that the mode is 13 and the median number is 7. The arithmetic mean is 8. Moreover, one of the whole numbers is precisely two less than the median.

Discussion: A strategy one might want to implement to solve this problem involves considering extreme cases. In this example, we know that the mode is 13. So, the extreme case would be that there are only two 13s (the worst-case scenario) and that the remaining numbers only appear once. Since one of the whole numbers was precisely two less than the median, we know that one number is 7-2=5. At this point, we know the numbers are: 5, 7, 13, 13. To obtain the remaining number, we utilize the arithmetic mean of 8 to get that the missing number is 2. Hence, the five numbers are 2, 5, 7, 13, and 13.

The following example (Example B) allows for varying points of view to solve.

Example B: Suppose 13 high schools in Oklahoma decide to have a volleyball tournament with each school having one team as their representative. This tournament is intense as it is a single-elimination tournament, meaning that the loser of each game is eliminated from the tournament. Since the coordinator only has one gym and one court available, they would like to know how many games will be played to obtain a winner.

Discussion: One method to solve the problem would be to map out the games and the winners. For example, since there are 13 teams, one could consider six distinct teams playing six different distinct teams with 1 team not playing. There will be six winners from the first round of games with 1 team yet to play. The six winners can then play each other with three distinct teams playing three different distinct teams. Again, the 1 team which has not played continues to wait its turn. We then take the three winners of the second round of games and include the 1 team who has not played. These four teams then play with two distinct teams playing two different distinct teams. We are down to 2 winners, and they play the final game. Overall, there will be 6+3+2+1 = 12 games to declare a winner.

Taking on a different point of view to solve Example B, one could consider the losers. Having one winner would mean there would need to be 12 losers. Hence 12 games are required to declare a winner. Ta-da!

An alternate view, also pointed out by Posamentier et al. (2015), would be to consider one team as a professional team that will definitely win each game. Then, consider the professional team playing each of the other teams. Since there are 12 non-professional teams, they will all lose to the professional team. Hence, 12 games are needed to declare a winner.

The following example (Example C) demonstrates the use of solving a similar analogous problem.

Example C: Determine the expression with the largest value without using a calculator.

1^{42}, 2^{35}, 3^{28}, 4^{21}, 5^{14}

Discussion: Students may try to compute these expressions by hand using brute force, or they may simply give up. If they could recognize a similar analogous problem that is more reasonable to solve, the situation would become less cumbersome. Notice that each exponent is a multiple of 7. Thus, if we take the 7^{th} root of each expression, we can simplify the terms by hand much more pleasantly and compare them. The largest original expression will be the one associated with the largest expression after taking the 7^{th} root. For example, after taking the 7^{th} roots of each term, the list becomes:

$1^6 = 1$, $2^5 = 32$, $3^4 = 81$, $4^3 = 64$, $5^2 = 25$.

Since $3^4 = 81$ is the largest number in the list, we can conclude that $(3^4)^7 = 3^{28}$ is the largest number in the original list.

Example D: A trapezoid has parallel sides of 5 and 7 units long. The trapezoid's two diagonals dissect the trapezoid's midline into three pieces. How long is each piece?

Discussion: This problem is intentionally ambiguous, as the only known measurements are the two parallel sides of a trapezoid. No information is given on whether the trapezoid is isosceles or not (it doesn't need to be) or its height. Students can use several problem-solving techniques to obtain a solution. For example, they could draw several trapezoids on graph paper and see how varying the angles does not affect the dimensions of the three segments. One can also experiment with other measurements to see if a greater difference in the size of the two parallel sides impacts the outcome (it does). Once a pattern is determined, students can conjecture an answer and then try to find a generalization to all trapezoids whose parallel sides are size *m* and *n* units long.

Example E: Consider the task of finding the derivative of a function. Examples of such functions are: $y = \frac{\cos x}{x^2}$ and $y = \frac{2x \sin x}{\tan x}$.

Discussion: This problem is a standard task, but it can be implemented in such a way that it supports sociomathematical norms in the classroom. For example, after students have studied various techniques to compute the derivative (power rule, product rule, quotient rule, trigonometric rules, and chain rule), they break into groups to find the derivative of a function similar to one given above. If there is enough time in class, the groups could be tasked to find the derivative in as many different ways as possible. There are various ways to compute the derivative as there are multiple representations of the same function (e.g. rewriting a quotient as a product, using trigonometric identities, etc.), which allows the students to simplify the function prior to computing the derivative using the appropriate rules. Or, students may compute the derivative of the function 'as is.'

Each group presents one method/solution to compute the derivative and if possible, they do not repeat a solution that has already been presented. Each time a group presents a solution, the class determines whether it is indeed a 'new' solution. Through these discussions, the students and instructor negotiate what constitutes *mathematical difference*, thereby cultivating sociomathematical norms in the classroom.

Example F: Consider an activity where students explore aspects of home loans with various interest rates and loan terms. A description of the activity is provided below; it can be found in its entirety at this link.

Description of Activity: This activity has two parts. After a short introduction on home loans by the instructor, the students begin the in-class part of the activity which focuses on the actual computations involved with home loans. Students are given the purchase price of a house, the percentage the buyer can afford as a down payment, the length of the term for the loan, and a fixed interest rate. Students are then asked to compute:

- The down payment
- The amount of the loan
- The amount of the monthly payment
- The total amount paid over the course of the loan
- The total amount of interest paid over the course of the loan.

The second part of the activity is to be completed out-of-class. This portion allows students the opportunity to create their own real-world problem based on their own personal interest and the research they conducted. Students begin by exploring a real estate website to find the selling price of various houses and lender websites to find the current mortgage rates. After students compute the monthly payment by hand, they are provided with detailed information on how to compute a monthly payment using a spreadsheet. They are then asked to compute the monthly payment required in various scenarios (different down payments, different interest rates, different loan terms) using the spreadsheet they created. After they complete the activity, they are provided prompts and are asked to reflect on their findings.

Discussion: This activity demonstrates various recommendations for pedagogical practices, such as fostering sociomathematical norms and incorporating the principles of curriculum design as established in the MIP components of inquiry.

The in-class portion of the activity is to be completed in groups of 3-4 students. One way to cultivate mathematical norms in the classroom is to give each student in the group a *role* as either a facilitator, recorder, or spokesperson.

The out-of-class portion of the activity is intended to be completed by each individual student as they will create their own unique problem that is meaningful to them. They will engage in problem solving by computing various monthly payments, primarily with the use of an Excel spreadsheet. Students will identify and find the necessary information required for the Excel formula and for various scenarios of their choice (i.e. Active Learning). Since students will be using technology for many of the computations, they can focus on the mathematical relationships between the various scenarios (i.e. Meaningful Applications). Moreover, the use of a spreadsheet provides an ease of computation which may reduce mathematical anxiety and increase confidence in their ability to do math (i.e. Academic Success Skills). Thus, in this activity, students will be engaged in all three MIP components of mathematical inquiry.

Where do we go from here? Creating Professional Learning Communities to Continue the Work

We encourage mathematics educators to create their own Professional Learning Communities (PLCs) to collaboratively enhance teaching practices and student outcomes. A PLC is a group of educators who work together to improve their skills and the academic performance of their students through continuous reflection and dialogue. Your PLC can focus on developing strategies to foster students' perseverance and engagement in productive struggle within entry-level post-secondary mathematics courses. This involves delving into the affective dimensions of mathematics learning, understanding the barriers and supports to student persistence, and recognizing the impact of our instructional practices. The following discussion questions are designed to guide these conversations, helping educators to reflect individually and collectively on how we can better support our students' positive affect and perseverance in mathematics.

Questions for Discussion

Understanding Students' Perseverance and Productive Struggle

1. **Defining Perseverance and Productive Struggle:**
   - How do we define perseverance and productive struggle in the context of our mathematics courses?
   - What does productive struggle look like in practice, and how can we recognize it in our students?
2. **Identifying Barriers:**
   - What are the common barriers that prevent students from persevering in challenging mathematical tasks?
   - How can we identify signs of frustration versus productive struggle in our students?
3. **Supporting Perseverance:**
   - What strategies have we found effective in encouraging students to persevere through difficult problems?
   - How can we create a classroom culture that values effort and persistence?

Promoting Engagement in Productive Struggle

4. **Instructional Practices:**
   - What instructional practices promote productive struggle in mathematics learning?
   - How can we design tasks that are appropriately challenging yet accessible for our students?

5. **Feedback and Assessment:**
    - What types of feedback encourage students to continue engaging with difficult problems?
    - How can our assessment practices support and recognize productive struggle?
6. **Role of Collaboration:**
    - How can collaborative learning activities support students in their productive struggle?
    - What role do peer interactions play in fostering perseverance in mathematics?

Affective Entailments of Mathematics Learning

7. **Emotional Responses:**
    - How do students' emotional responses to challenges impact their perseverance and engagement?
    - What strategies can we use to help students manage their emotions to access a positive affective cycle during challenging tasks?
8. **Building Confidence:**
    - How can we help students build confidence in their ability to tackle difficult mathematical problems?
    - What role does self-efficacy play in students' perseverance and how can we nurture it?

Instructor Agency and Reflection

9. **Instructor Influence:**
    - How do our attitudes and beliefs about perseverance and struggle influence our teaching practices?
    - In what ways can we model perseverance and a positive attitude towards struggle for our students?
10. **Reflective Practices:**
    - How can we use reflective practices to improve our support for students' perseverance and productive struggle?
    - What personal experiences with struggle and perseverance can we share with our students to inspire them?

Collective Strategies and Community Building

11. **Sharing Best Practices:**
    - What are some successful strategies we've implemented that have helped students persevere?
    - How can we share these practices and learn from each other's experiences within our PLC?
12. **Creating a Supportive Environment:**
    - How can we collectively create a supportive environment that encourages risk-taking and resilience?
    - What resources (e.g., articles, tools, workshops) can we utilize to further develop our understanding and practices related to perseverance and productive struggle?

Closing the Loop

13. **Continuous Improvement:**
    - How can we assess the effectiveness of our strategies in promoting perseverance and productive struggle?
    - What ongoing professional development opportunities can help us stay informed about best practices in this area?

Conclusion

Mathematics often presents challenges that can hinder students from developing a positive identity as math learners. However, teachers play a crucial role in guiding students from a negative affective path to a positive affective cycle. This paper explores actionable strategies for transforming the affective elements of learning by implementing the MIP Guiding Principles—a practical framework for creating a supportive classroom environment and nurturing students' emotional relationships with math. Through these principles, educators foster engagement, promote perseverance, and create an atmosphere that embraces productive struggle.

Actively engaging students and demonstrating the various connections in math can boost their confidence, reduce anxiety, and make math more approachable. Meaningful applications and active learning may enhance motivation, helping students recognize the value of math beyond the classroom. By designing engaging activities, including ill-structured tasks, teachers can encourage students to persevere and develop effective problem-solving skills. Collectively, these practices aim to enrich students' emotional connection to math, promoting a more positive and productive learning experience. Through these approaches, students are likely to become more engaged, develop a stronger math identity, and achieve greater success.

Glossary of Terms:

- *Affect*—the psychological experience of emotions, moods, or feelings.

- *Attribution Theory*— examines how people explain their successes and failures, differentiating between internal factors (influences within themselves) and external factors (influences outside their control).

- *Cognition*—the mental processes of acquiring and understanding knowledge through thought, experience, and the senses.

- *Ill-structured exercise/activity*—a problem or exercise with some level of built-in ambiguity and no prescribed path for a solution. The students must reason through what is given and what, if anything, must be assumed.

- *Perseverance*—Tenacity in doing something challenging despite a delay in achieving success.

- *Plug-and-Chug*—A process where numerical values are entered into a formula to obtain an answer without needing problem-solving skills.```

- *Problem solving*—The action one takes to achieve an outcome in which the individual has no known method to utilize (Schoenfeld 1985, 2013).

- *Productive Struggle*—The necessary component of learning that involves grappling with perplexing or unfamiliar ideas until a break-through occurs.

- *Professional Learning Community (PLC)* —A PLC is a group of educators who work together to improve their skills and the academic performance of their students through continuous reflection and dialogue.

- *Sociomathematical Norms*—The established guidelines for how students and instructors should interact in a mathematics classroom, particularly related to how mathematical ideas are developed and justified.

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