Lesson 3

TITLE OF LESSON: Combining Vertical and Horizontal Transformations for Quadratic Equations

ESTIMATED TIME FOR LESSON (IN MINUTES): 50-75 minutes

SUGGESTED FORMAT (check all that are appropriate):

Individual in-class
Collaborative in-class
Individual homework
Collaborative homework

OVERVIEW:

- Students use Desmos to investigate graphs of general quadratic functions.
- Students develop a set of rules for using transformations to derive a graph from parameter values when the function is written in a particular formulation.
- Students investigate alternate formulations and how these affect the rules for deriving an accurate graph using transformations.

PREREQUISITE IDEAS AND SKILLS:

- Students should be able to multiply (distribute) a constant across a binomial.
- Students should be able to factor a constant out of a binomial.
- Students should be familiar with the order of operations.
- Students should be familiar with the Cartesian coordinate system.
- Students should understand what a function is and the different ways it can be represented (e.g., graphs, equations).
- Students should understand what a transformation is.
- Students should be able to describe single vertical transformations (shifting, stretching, reflecting) applied to the parent function $f(x) = x^2$, both algebraically and graphically, as covered in Lesson 1.
- Students should have completed Lesson 2 before attempting this lesson.

MATERIALS NEEDED TO CARRY OUT LESSON:

- Instructor access to Internet for instructor demonstrations
- Students access to Internet for students' collaborative exploration
- Example Formulations Key for instructor.

CONCEPTS TO BE LEARNED/APPLIED:

Guided exploration will help students to:

- Use shifting and stretching/reflecting to graph any quadratic function, when given in a suitable format. (Functions given in standard form will need to be reformulated first, a skill not covered here.)
- Articulate a set of rules for interpreting parameter values as horizontal shifts, vertical shifts, horizontal or vertical stretches or compressions, and horizontal or vertical reflections, including in which order to apply transformations when more than one are present.

ACTIVITIES:

ACTIVITY 1: Incorporating a Horizontal Shift (20 minutes)

Goals:

- Students consider how to incorporate horizontal shifts by adding a new parameter to their formulation from Lesson 2.

Steps:

Note: These instructions are a little abstract, since the authors don't know which formulation you've adopted as your official class formulation. To help clarify, the "Example Formulations Key" takes you through the steps using specific formulations as examples.

Creating Horizontal Effects. Present students with whichever formulation you've chosen as your official class formulation for a quadratic function with no x-term (from the previous lesson plan, Lesson 2). Remind them of their set of rules for turning parameter values into an accurate graph of the function.

Ask the following questions:

- Can we graph every possible parabola using this formulation?
  (No.)
- What parabolas are we unable to graph?
  (Any parabola with a vertex not on the y-axis.)
- Describe a transformation that would allow us to graph one of the "missing" parabolas.
  (A horizontal shift or translation would be needed. Speaking informally, we need to be able to shift our parabolas to the left or right.)
- Would the addition of this transformation allow us to graph any possible parabola?
  (Yes.)
- Can we express horizontal transformations using the parameters we already have in our formulation?
  (No.)
- Are we allowed to throw in a new parameter to represent a horizontal shift?
  (Yes - it's *our* formulation! Have students choose a letter for their new parameter.)

Horizontal Shift Desmos Exercise

- Open a new Desmos graph and have students do the same.
- Show students how to type your official class formulation for a quadratic function with no x-term (from the previous lesson plan, Lesson 2). into the first row. Click "All" to add sliders for your two parameters.
- Challenge students to add a new parameter (along with its corresponding slider) somewhere in your formulation in order to affect a horizontal transformation.
- In other words, moving their new slider should do nothing to the graph other than shift it left or right.
- Give students a few minutes to attempt this feat. If they need hints, refer to this list:

  Hints about where to place the parameter for a horizontal shift:
  (Work slowly down this list as needed until students can make progress on their own.)
  - If it isn't obvious from your formulation, manipulate it so that students can observe the two parameters acting on the y-value.
  - Point out that this is why the parameters create vertical effects; since y is the vertical axis, multiplying or adding to the y-value will create a *vertical* effect.
  - So our new *horizontal* shift parameter should change… not the y-value… but the… x-value, yes!
  - If students try to put the new parameter outside the x^2, as in $x^2 + b$, manipulate their attempt on paper to show that they are still only affecting the y-value.
  - And/or make the point that exponents are evaluated first, so the x-value (input) is immediately squared, which turns it into a y-value (output) before the parameter can have any effect. Thus the parameter will only affect the y-value, or output of the parent function, and will only have a vertical effect on the graph.
  - Put a set of parentheses around the x, as in: $(x)^2$, and state that we have the option to place a parameter inside those parentheses.

Conclusion

Once a majority of students have successfully added a horizontal shift to their formulation, re-group for the following discussion:

- How did you add your new parameter to the formulation?
  (The parameter should be added to or subtracted from the x variable inside parentheses, with the exponent 2 located outside the parentheses.)
- Does it do nothing but create a horizontal shift?
  (The answer should be yes; if a student answers no, encourage them to try a different solution.)
- How does this alter our set of rules for graphing a quadratic function using transformations?
  (Student answers may vary. A new rule should be added which moves the parabola left or right, depending on whether the new parameter is added or subtracted. Make sure to examine each possibility.)

- Does it surprise you that the parabola moves in the direction it does?
  (Students will likely be surprised to find that adding a positive value to the input moves the parabola to the left while adding a negative value moves the parabola to the right.)
- Can we explain this surprising result – why does it work that way?
  (Answers may vary. When we add to the input before applying the parent function, we will obtain the output from a value to the right of our current location, thus "pulling" the parabola to the left.)
- Can we now graph any quadratic function using transformations?
  (Yes!)

If you plan to continue with Activity 2, ask the following questions as a lead-in:

- What transformations are we using?
  (The list should include: vertical shift (or translation), vertical stretching or shrinking, vertical reflection, and horizontal shift (or translation).
- Are there any transformations we've left out? What are they?
  (Yes, horizontal stretching or shrinking and horizontal reflection. You may need to hint at this by stressing the word "vertical" in the above list.)
- Do we ***need*** to include horizontal stretching/shrinking and horizontal reflection?
  (See below to lead this discussion…)

<u>ACTIVITY 2: Incorporating Horizontal Stretch/Shrink and Reflection (40 minutes)</u>

Goals:

- Students consider how to incorporate horizontal stretching/shrinking and reflection by adding a second new parameter to their formulation from Activity 1 above.

Steps:

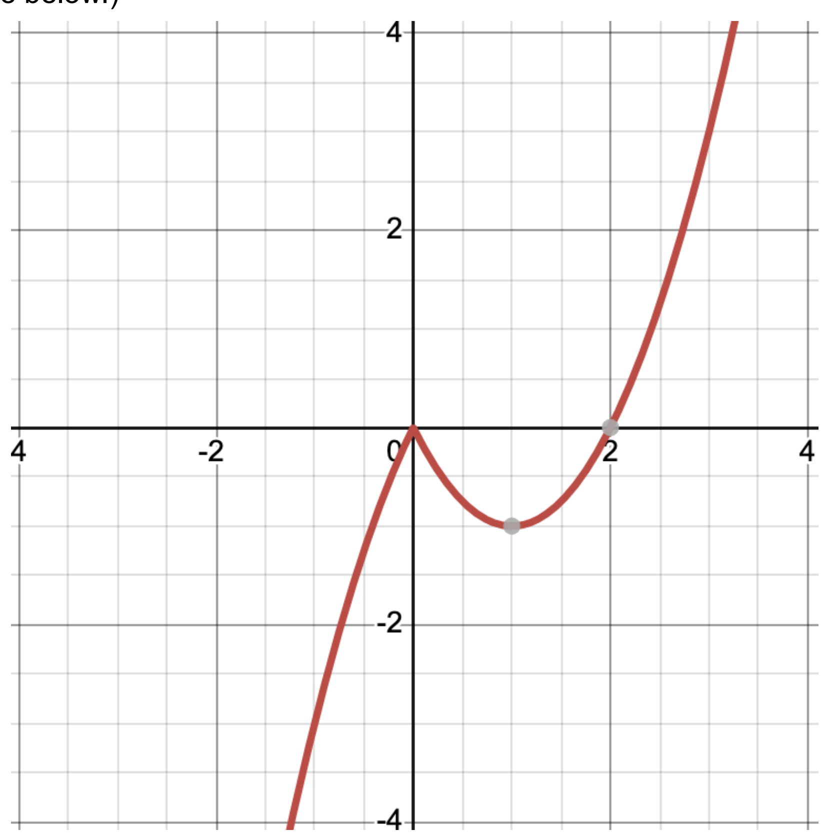
Discussion on Horizontal Stretching/Shrinking and Reflection
(Optional) To motivate this activity, we recommend discussing the need for this new parameter as follows:

- Do we need another parameter in order to graph every possible parabola?
  (No. We can graph every possible parabola with the parameters and corresponding transformations we're already using.)
- How are we able to stretch a parabola horizontally if we don't have a parameter for that?
  (For a parabola, a horizontal stretch is equivalent to a vertical squeeze, which we do have a parameter for.)
- How are we able to reflect a parabola horizontally if we don't have a parameter for that?
  (Since a parabola has horizontal symmetry, a horizontal reflection about the parabola's axis of symmetry makes no difference. A horizontal reflection about the origin creates the same effect as a horizontal shift, and we do have a parameter for that.)

- Keeping in mind that our ultimate goal is to use transformations to graph different types of functions, is there any good reason to include a parameter that represents our missing transformations, horizontal stretching/shrinking and horizontal reflecting?
  (See what students have to say here, then fill in the gaps from the list below.)

  Reasons for including a fourth parameter
  - It would give you the ability to directly express the idea of a horizontal stretch or reflection.
  - If a formula were given to us in a way that makes horizontal stretching and reflecting obvious, we would want to recognize that so we wouldn't have to re-formulate before graphing.
  - Most importantly, even though parabolas do not require horizontal stretching/shrinking and reflecting, other functions will.

Bonus Question
- Can we think of a function shape that would require all the different transformations to be able to graph all possible variations of that shape?
  (Many such shapes exist. To make sure horizontal reflections are required, the shape should not have horizontal symmetry or $180°$ rotational symmetry. To make sure both horizontal and vertical stretching is required, the shape should have at least two critical points that do not share x or y values. These details are not important for students to know right now, but are included to assist instructors in finding examples. See the example below.)



Desmos Activity

- Have students open a new Desmos graph and input the quadratic formulation we have so far from Activity 1, along with sliders for each parameter.
- Instruct students to figure out where to add a fourth parameter so that it will only affect horizontal stretching/compressing and horizontal reflection.
- Ask students to then write down a set of rules to go with their new formulation. Rules should include the following:
  - Which parameters cause a vertical stretch/compression, reflection or shift and which cause a horizontal stretch/compression, reflection or shift?
  - Where is the axis located for each reflection?
  - What is the direction and size of each shift? How do we know whether a shift goes up or down / left or right? By how many units?
  - For each stretch or compression, by what factor?
  - The order in which to apply these transformations.