Lesson 2

TITLE OF LESSON: Combining Vertical Transformations for Quadratic Equations

ESTIMATED TIME FOR LESSON (IN MINUTES): 50-75 minutes

SUGGESTED FORMAT (check all that are appropriate):

Individual in-class
Collaborative in-class
Individual homework
Collaborative homework

OVERVIEW:

- Students use an interactive Desmos applet to investigate graphs of functions of the form $f(x) = ax^2 + c$.
- Students develop a set of rules for deriving a graph directly from the values of a and c.
- Students investigate alternate formulations of functions of this type and how these formulations might affect their set of rules.

PREREQUISITE IDEAS AND SKILLS:

- Students should be able to multiply (distribute) a constant across a binomial.
- Students should be able to factor a constant out of a binomial.
- Students should be familiar with the order of operations.
- Students should be familiar with the Cartesian coordinate system.
- Students should understand what a function is and the different ways it can be represented (e.g., graphs, equations).
- Students should understand what a transformation is.
- Students should be able to describe single vertical transformations (shifting, stretching, reflecting) applied to the parent function $f(x) = x^2$, both algebraically and graphically, as covered in Lesson 1.

MATERIALS NEEDED TO CARRY OUT LESSON:

- Instructor access to Internet for instructor demonstrations
- Students access to Internet for students' collaborative exploration
- "Alternate Formulations" Worksheet

CONCEPTS TO BE LEARNED/APPLIED:

Guided exploration will help students to:

- Use shifting and stretching/reflecting to graph any quadratic function given in the form $f(x) = ax^2 + c$.

- Articulate a set of rules for interpreting the a and c values as vertical shifts, stretches, and reflections, including in which order to apply transformations when more than one are present.

ACTIVITIES:

ACTIVITY 1: Introduction (20 minutes)

Goals:

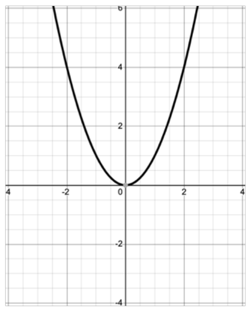
- Students discover that when two vertical transformations are applied graphically to a parent function,
  - the order in which they are applied makes a difference to the resulting graph.
  - the way "reflection" is applied makes a difference to the resulting graph.
- Students discover algebraic manipulation of a function's formula as a useful tool since
  - equivalent equations have equivalent graphs.
  - equivalent equations allow students to think about the graphing rules being applied in different ways (e.g., applying the "rules" in a different order).

Steps:

An Example of Two Transformations. Present students with the function: $f(x) = -x^2 + 3$. Talk them through graphing this function by using the transformations we learned in the previous lesson. Ask the following sequence of questions in order.

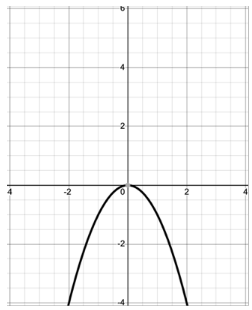
- What is the parent function?
  ($f(x) = x^2$)
- What does the graph of the parent function look like?
  (An upward-facing parabola with its vertex at the origin. Graph the parent function.)
- What should the plus 3 do to this graph?
  (Shift it three units up.)
- What should the negative sign do to this graph?
  (Reflect it, or flip it upside down.)
- Combining these two actions, what will the resulting graph look like? Sketch the graph on your own paper, then describe it to me. Tell me how you arrived at your graph.
  (Answers may vary at this point; no corrections needed.)
- Collect student suggestions and illustrate the following possibilities on the board, then ask which is correct? Ask students to justify their answers. Accept any justifications at this point. The graphic below may be helpful.
  - If we move the graph up three then reflect over the x axis, we get a downward-facing parabola with vertex at (0,-3).
  - If we reflect over the x-axis, then shift the graph up three, we get a downward-facing parabola with vertex at (0,3).
  - If we move the graph up three, then reflect vertically over the vertex, we get a downward-facing parabola with vertex at (0,3).

$f(x) = x^2$



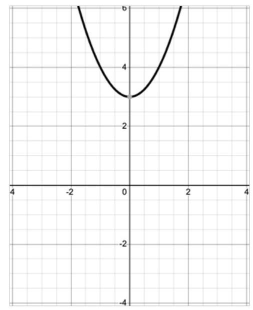
Parent Function
(No Transformation)

$f(x) = -x^2$



Vertical Reflection

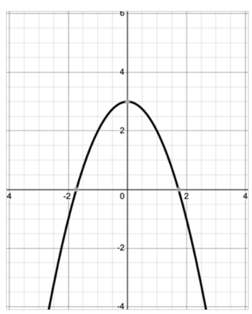
$f(x) = x^2 + 3$



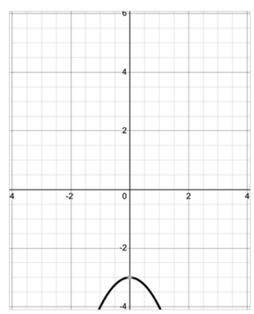
Shift (or Translation) up by 3

$$f(x) = -x^2 + 3?$$

Which graph is correct for a vertical reflection combined with an upward shift?



Reflect first, shift second (or shift, then reflect over vertex.)



Shift first, then reflect over the x-axis.

- We're getting two different results. How can we know which is correct?
  (One way to know is to plot some points. Use $x = 0$ and maybe another x-value as input to see which graph includes the plotted points. We can also think about how the output value is affected. Using order of operations, it's clear each input will have its sign reversed before adding three; thus it makes sense that the graph should be reflected before it gets shifted.)
- What were the differences in our approaches that led to two different graphs?
  (It matters whether you reflect first and then shift, or shift first then reflect.
  It also matters whether you reflect over the x-axis or reflect vertically over the vertex point.)

Conclude by stating that your challenge for the class today is for them to become confident at determining the correct graph from a formula by using transformations. (So that you don't have to plot points, except maybe to check your work.) Tell them before we get started, let's take a closer look at the formulas we'll be using….

Alternate Formulations. Ask students to look over these example formulations (taken from the "Alternate Formulations" worksheet):

$$f(x) = a(x^2 - c) \qquad y = mx^2 + n$$

$$f(x) = bx^2 - d \qquad y - g = hx^2$$

$$f(x) = q(x^2 + p) \qquad x^2 = j(y - k)$$

$$f(x) = ex^2 + f \qquad x^2 = wy + r$$

Ask the following questions:

- What do these equations all have in common?
  (All include an equal sign, an x^2 term, a y or $f(x)$ term, and two parameters, one that's multiplied, and one that's added or subtracted. These patterns may not be easy for students to see at first. Get them thinking about it by asking them to name or describe individual elements in each formulation. Remind them that by convention, x is an independent variable, y or $f(x)$ represent a dependent variable, and other letters represent constants, or parameters.)
- How do the equations differ?
  (Some are written as functions using $f(x)$, some not; some have parentheses; positives and negatives appear in different places; varying elements are located to the left of the equal sign.)

Hand out a copy of the "Alternate Formulations" Worksheet to each student. Step students through the first part of the worksheet by recreating the table at the top as follows:

- Write these two examples on the board:
  - $f(x) = 6x^2 + 12$
  - $f(x) = 6(x^2 + 2)$
- Ask whether these are the same function. (Yes!)
- Show that they are the same function by simplifying the second example.
- Write these two formulations next to the original two examples:
  - $f(x) = ax^2 + c$
  - $f(x) = a(x^2 + c)$
- State that these are two different "formulations" for what a quadratic function looks like when it doesn't have an x term. What makes them different? (The set of parentheses.)
- If I changed $f(x) = ax^2 + c$ to $f(x) = bx^2 + d$, would that create a *different* formulation? (No - selecting different letters for the parameters does not change a formulation to a different formulation.)
- Point out that each formulation includes two parameters (a and c). To indicate a specific function, the parameters will take on specific numbers as values.
- Ask students whether the same set of rules work for both formulations in the table. For example, if I said, "Stretch the parent function by a factor of a, then shift up c units" would that lead to the same graph for each of these formulations?

(The answer is no, and you can illustrate this by pointing back to the original examples. In Example One, the value of c is 12, and in Example Two, the value of c is -2. Thus following the same set of rules for each formulation yields very different results.)

- Conclude by stating: The worksheet will ask you to come up with a set of rules for graphing these functions. Your rules will include a formulation of your choice, and the formulation you choose will determine the specifics of your set of rules.

<u>ACTIVITY 2: Worksheet/Discussion (20 minutes)</u>

Goals:

- Students use Part 2 of the "Alternate Formulations" worksheet and the Desmos app to come up with a set of rules for correctly graphing quadratic functions with no x-term using transformations.
- The class adopts an official formulation and set of rules.

Steps:

Worksheet. Show the class how to enter a formulation into Desmos, creating sliders for each of the two parameters. Give students time to work through the worksheet. Circulate and support with clarifications, hints, or suggestions. Encourage students to test their results by plotting points (especially when you notice them getting it wrong or if they ask you to validate their answers). Remember this can be done mentally; a quick, estimated calculation to check whether a point likely lies on the graph is often sufficient to reveal a graph as being incorrect.

Discussion.

- Collect students' chosen formulations on the board.
- Write student-supplied rules below each formulation.
- Test the rules using examples from the worksheet and correct as needed.
- Either vote on an official class formulation with a set of rules or state which one is official according to your text.
- Use the following examples to test your newly adopted set of rules:
    - Re-formulate each example into your chosen formulation.
    - Follow the rules to sketch a graph using transformations.
    - Check your results by graphing each example on Desmos.
    - Examples:
      1. $f(x) = 2x^2 - 3$
      2. $x^2 = .3y + 4$
      3. $y = -(x^2 + 4)$

Shifting, Stretching, Reflecting Conclusion. Ask the following questions to set up the next lesson:

- Can we now graph any given quadratic function?
  (No - we haven't discussed how to graph a function with an x-term.)
- What parabolas are we able to graph so far?
  (Any parabola whose vertex is on the y-axis.)
- What parabolas are we not yet able to graph?
  (Any parabola whose vertex lies off the y-axis.)

Conclude with the statement that next time we'll look at how to add horizontal shifts, stretches, and reflections to our graphing abilities.