Answers are provided in blue

**Activity 1 - Shapeshifting**

**Part I -** Consider the following functions, F, G, H, and K.

* F takes an input of ⛋ and yields an output of ☐,
* G takes an input ✩ of and yields an output of ⛋,
* H takes an input of ⛋ and yields an output of ◯, and
* K takes an input of ♡ and yields an output of ☐.

F: ⛋ ↦ ☐

G: ✩ ↦⛋

H: ⛋ ↦ ◯

K: ♡ ↦ ☐

**a.** Chloe has a ✩, but wants a ☐. Can she get what she wants? If so, explain how and specify which functions will be used and in what order.

Yes, use G to get ⛋, then F to get ☐

**b.** James has♡, but wants a ◯. Can he get what he wants? If so, explain how and specify which functions will be used and in what order.

No, no functions give take ☐

**c.** Kierra has a ⛋, and wants to use H and K to get a ☐. Will this work? Why or why not?

No, H will give ◯ and not other functions take ◯.

**Part II** - Consider the following relations.

**A:**

| **Input** | ◯ | ⦻ | ⦵ | ⏀ |
| --- | --- | --- | --- | --- |
| **Output** | ✩ | △ | ◯ | ◇ |

**B:**

| **Input** | ☐ | ◯ | ◇ | △ | ♡ |
| --- | --- | --- | --- | --- | --- |
| **Output** | ⦵ | ⦻ | ⛋ | ♡ | ✩ |

1. Do the above relations represent functions? Why or why not?
2. Who can get what they want?
	1. Chloe has a ☐, but wants a ◯. Can she get what she wants? If so, explain how and specify which functions will be used and in what order.

 Yes: Use B, then A in that order.

B(☐)= ⦵, A(⦵) = ◯

* 1. James has⏀, but wants a ♡. Can he get what he wants? If so, explain how and specify which functions will be used and in what order.

No, ⏀ only goes to ◇ and ◇ only goes to ⛋.

No other options exist.

* 1. Jolynn has◯, but wants a ♡. Can he get what he wants? If so, explain how and specify which functions will be used and in what order.

Yes: B(◯) = ⦻, A(⦻) = △, B(△) = ♡,

First use, B, then A, then B.

* 1. If you have ☐, what all the possible shapes could you produce?

 ⦵, ◯, ⦻,△,♡,✩

1. If we use function notation to write A( ◯ ) = ✩, A( ⦻ ) = △, etc, describe the solution to (a)-(c) above using function notation.
2. A(B(☐)) = ♡,
3. No, ⏀only goes to ◇ and ◇ only goes to ⛋. No other options exist.
4. Yes: Use B, A, and A in that order.

B(A(B(◯)))=♡

**Activity 2 - International Business Deal**

You are the owner of a medium-sized manufacturing company based in the United States. You've been working on expanding your business to cater to international clients. An opportunity arises to work with a client in Brazil who wants to purchase a large shipment of your products.

**Negotiation and Agreement:** After negotiations, you and the Brazilian client agree on a deal worth **1.2 million Brazilian reais (BRL)**. The Brazilian client prefers to pay in their local currency, BRL. However, your company operates primarily in US dollars (USD), so you need to convert the payment to your home currency.

1. If the current exchange rate is **1 USD = 5 BRL**, how much is this deal worth in US dollars (USD)? If it helps, you can use the conversions:
* $B(u) = 5u$ to change ***u*** USD to BRL
* $U(b) = \frac{1}{5}b$ to convert ***b*** BRL to USD

1,200,000 BRL (1 USD / 5 BRL) = 240,000 USD

**Purchase Supplies:** To fulfill the order, you need to buy specialized parts. You have found two possible suppliers:

**Supplier 1:** One supplier is in Germany, which only accepts payment in euros (EUR). Production and shipping of the parts from Germany cost **95,000 EUR**.

**Supplier 2:** One supplier is in China, which only accepts payment in Chinese Yuan (CNY). Production and shipping of the parts from China cost **493,000 CNY**.

To file international taxes, you need to invoice these costs in Brazilian reais (BRL).

1. Find the total cost of parts and services from each supplier in Brazilian reais (BRL). For each step, describe what functions you use in what order.

Current exchange rates are:

* **1 USD = 5 BRL**
	+ $B(u) = 5U$ to convert ***u*** USD to BRL
	+ $U(b) = \frac{1}{5}B$ to convert ***b*** USD to USD
* **1 USD = 0.95 EUR, i.e.**
	+ $E(u) = 0.95U$ to convert ***u*** USD to EUR
	+ $U(e) = \frac{1}{0.95}E$ to convert ***e*** EUR to USD
* **1 USD = 7.25 CNY, i.e.**
	+ $C(u) = 7.25U$ to convert ***u*** USD to CNY
	+ $U(c) = \frac{1}{7.25}E$ to convert ***c*** CNY to USD

**German Parts - first U(e), then B(u)**

95,000 ( 1USD/0.95 EUR) = 100,000 USD

100,000 ( 5 BRL/ 1 USD) **= 500,000 BRL**

 **Chinese Parts - first U(c ), then B(u)**

 493,000 CNY (1 USD / 7.25 CNY) **= 68,000 USD**

 68,000 USD (5 BRL / 1 USD) = **340,000** **BRL**

**Cost of Services:** In researching both orders, you find that the German parts come with built-in monitoring technology, while the Chinese parts don’t. This being the case, the installation will cost **110,000 USD** using the German parts, versus **122,000 USD** for the Chinese parts.

1. Again, for invoicing, you need to record these costs in Brazilian Reais (BRL). What would be the costs for each installation in BRL?

 Installation using German parts

 110,000 USD ( 5 BRL/1 USD) = **550,000 BRL**

 installation using Chinese parts

122,000 USD ( 5 BRL/1 USD) = **610,000 BRL**

1. What would be the total profit from each deal, in Brazilian Reais (BRL)? Which supplier would you use?

 **Profit using German parts**

 1,200,000 - 550,000 - 500,000 = **150,000 BRL**

 **Profit using Chinese par**t**s**

1,200,000 - 340,000 - 610,000 = **250,000 BRL**

1. If you pay dual corporate tax on profits, 21% tax in the United States and 34% tax in Brazil, how much profit would you make on this deal after taxes (net income, or net profit after tax)? What is your after-tax profit margin (percent after-tax profit of total deal)?
*Optional:* How does this after-tax profit margin compare to the profit margin using the other supplier, say, for example, if new tariff laws excluded trade with the more profitable suppliers?

Students can approach this in a number of different sequences. Part of the purpose of this is to allow them to try and discuss alternative approaches which should yield the same answers in this case.

**Chinese Parts - Post-tax Profit**

 250,000 BRL (0.55 untaxed) = 137,500 BRL ( 1 USD / 5 BRL ) = **27,500 USD**

**Chinese Parts - Post-tax profit margin**

27,500/240,000 USD = 0.102083 = **11.458%**

**German Parts - Post-tax Profit**

 150,000 BRL (0.55 untaxed) = 82,500 BRL ( 1 USD / 5 BRL ) = **16,500 USD**

**German Parts Post-tax profit margin**

16,500/240,000 USD = 0.102083 = **6.875%**

*These scenarios demonstrate how analysis of international business deals can often involve multiple conversions which can be efficiently calculated using function composition. The next series of activities develops skills for numerically evaluating such composition.*

**Activity 3 - Numerical Function Composition**

At this point, we’ve developed an understanding of function composition, and, using visual examples, we have seen how certain function compositions cannot be constructed. The reasoning used in the visual examples of Activity 1 (that involved a comparison of inputs and outputs) can be applied to functions expressed in their usual forms: algebraically, graphically, and numerically. In this activity, we will use functions represented numerically to explore when functions can and cannot be constructed.

**Part I-** Consider f(x) and g(x) below.

| $x$ | 1 | 3 | 9 | 12 | 18 | 29 | 34 | 45 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ | -3.5 | 6 | 78 | -21 | 20.3 | 16 | 27 | 35 |

$$

| $x$ | 5 | 8 | 19 | 22 | 28 | 35 | 37 | 48 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $g(x)$ | 1 | 18 | 9 | 34 | 3 | 29 | 12 | 45 |

1. Find $f(g(5))$ -3.5
2. Find $f(g(8))$ 20.3
3. Find $f(g(19))$ 78
4. Find $f(g(22))$ 27
5. Find $f(g(28))$ 6
6. Find $f(g(35))$ 16
7. Find $f(g(37))$ -21
8. Find $f(g(48))$ 35
9. What is the domain of $(f∘g)(x)$? {5, 8, 19, 22, 28, 35, 37, 48}
10. Find x such that $f(g(x)) = 27$
11. Find x such that $f(g(x)) = 35$
12. Use the table below to construct the new function, $(f∘g)(x)$.

| $x$ | 5 | 8 | 19 | 22 | 28 | 35 | 37 | 48 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $(f∘g)(x)$ | -3.5 | 20.3 | 78 | 27 | 6 | 16 | -21 | 35 |

**Part II-** Consider $f(x)$ and $h(x)$ below.

| $x$ | 1 | 3 | 9 | 12 | 18 | 29 | 34 | 45 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ | -3.5 | 6 | 78 | -21 | 20.3 | 16 | 27 | 35 |

| $x$ | -1 | 0 | 6 | 11 | 15 | 18 | 25 | 31 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $h(x)$ | 0 | 2 | -8 | 15 | 28 | -30 | -45 | 56 |

1. For what x value can we find an output for $f(h(x))$? None
2. What can we conclude about the composition $(f∘h)(x)$? It does not exist

**Part III-** Consider $f(x)$ and $k(x)$ below.

| $x$ | 1 | 3 | 9 | 12 | 18 | 29 | 34 | 45 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ | -3.5 | 6 | 78 | -21 | 20.3 | 16 | 27 | 35 |

| $x$ | -4 | -2 | 5 | 9 | 12 | 20 | 34 | 40 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $k(x)$ | 3 | -5 | 12 | 0 | 1 | -34 | 3 | 21 |

1. Find $f(k(-4))$ 6
2. Find $f(k(-2))$ DNE
3. Find $f(k(5))$ -21
4. Find $f(k(9))$ DNE
5. Find $f(k(12))$ -3.5
6. Find $f(k(20))$ DNE
7. Find $f(k(34))$ 6
8. Find $f(k(40))$ DNE
9. What is the domain of $(f∘k)(x)$? What is the range of $(f∘k)(x)$? D: {-4, 5, 12, 34} R: {6, -21, -3.5}
10. Solve for x in $f(k(x))= -21$ x = 5
11. Solve for x in $f(k(x))= -3.5 $x = 12
12. Use the table below to construct a new function $(f∘k)(x)$. (Note that not all columns may be used)

| x | -4 | 5 | 12 | 34 |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| f(k(x)) | 6 | -21 | -3.5 | 6 |  |  |  |  |

1. Using the newly recreated table, double-check your answers from parts j and k.

**Activity 4 - Numerical Function Composition with Units**

In addition to there being complications with function composition due to the numerical values involved, there can also be issues that arise due to the context in which the functions are set. In this activity, you will have to consider both the numerical and the contextual as you work with the provided functions and their composition.

**Part I -** Consider $f(x)$ and $g(x)$ below, which track some of your “incredible” exercising and dieting trends.

| $x$ calories burned | 10 | 13 | 29 | 35 | 48 | 59 | 64 | 75 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ grams of sugar you will let yourself eat | 3.5 | 6 | 18 | 21 | 29 | 45 | 66 | 85 |

| $x$ minutes exercising | 5 | 8 | 19 | 22 | 28 | 35 | 37 | 48 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $g(x)$ calories burned | 10 | 13 | 29 | 35 | 48 | 59 | 64 | 75 |

1. Consider $(f∘g)(x)$ numerically, without taking the units into consideration. Does this composition make sense? If so, what is the domain? What is the range?

Yes. The range of $g(x)$ is has values in the domain of $f(x)$. D: {5, 8, 19, 22, 28, 35, 37, 48} R: {3.5, 6, 18, 21, 29, 45, 66, 85}

1. Now, take the units into consideration. Can we make sense of $(f∘g)(x)$? If so, build a new table making sure to include units.

 Yes.

| $x$ minutes exercising | 5 | 8 | 19 | 22 | 28 | 35 | 37 | 48 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ($f∘g)(x)$ grams of sugar you will let yourself eat | 3.5 | 6 | 18 | 21 | 29 | 45 | 66 | 85 |

1. What composition should you do if you wanted to know how much sugar you would eat if you had exercised for 22 minutes? What is the amount of sugar?

 f(g(22)) = 21

1. What equation would you be solving if you wanted to find the amount of exercise you would have to if you wanted to consume 45 grams of sugar? What is the amount of minutes you must exercise?

 f(g(x)) = 45 x = 35 minutes of exercise

**Part II-** Consider $f(x)$ and $k(x)$ below in which $f(x)$ is a function of one’s profit after having been open for $x$ days and $k(x)$ is the number of packages (of iPads) sold after spending $x$ hundred dollars in marketing.

| $x$ days since opening | 1 | 3 | 9 | 12 | 18 | 29 | 34 | 45 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ hundred dollars in profit | 3.5 | 6 | 78 | 21 | 20.3 | 16 | 27 | 35 |

| $x$ hundred dollars in cost | 1 | 2 | 5 | 9 | 12 | 20 | 34 | 40 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $k(x)$ packages sold | 1 | 3 | 9 | 12 | 18 | 29 | 34 | 45 |

1. If we ignore the units, can we make sense of $(f∘k)(x)$? Yes.
2. If we account for the units, can we make sense of $(f∘k)(x)$? If not, why? No. The units of “packages sold” do not align with inputs of “days since opening”

**Part III-** In the following two exercises, we will be looking at tables for which the input and output values present no barrier to the existence of either $(f∘g)(x)$ or $(g∘f)(x)$. These problems will prepare us for the following and final activity in which we work with functions represented by words, not tables.

| $x$ dollars spent on coffee | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ cups of coffee to drink | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

| $x$ cups of coffee to drink | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $g(x)$ trips to the restroom | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

1. Does $(f∘g)(x)$ exist? Explain. If the composition does exist, provide the input units and output units for $(f∘g)(x)$. No. The units for the range of $g(x)$ does not align with the units for the domain of $f(x)$.
2. Does $(g∘f)(x)$ exist? Explain. If the composition does exist, provide the input units and output units for $(g∘f)(x)$. Yes. The numerical values for the range of $f(x)$ are the same as the numerical values in the domain of $g(x)$, and the units for the range of $f(x)$ aligns with the units for the domain of $g(x)$. The input units are $x$ dollars and the output units are $g(f(x))$ number of trips.

| $x$ apple trees planted | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $f(x)$ square meters of shade grass needed to plant | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

| $x$ hundred pounds of apples requested | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $g(x)$ apple trees needed to plant | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

1. Does $(f∘g)(x)$ exist? Explain. If the composition does exist, provide the input units and output units for $(f∘g)(x)$. Yes. The numerical values for the range of $g(x)$ are the same as the numerical values in the domain of $f(x)$, and the units for the range of $g(x)$ aligns with the units for the domain of $f(x)$. The units for the input are $x$ hundred pounds and the units for the output are $f(g(x))$ square meters.
2. Does $(g∘f)(x)$ exist? Explain. If the composition does exist, provide the input units and output units for $(g∘f)(x)$. No. The units for the range of $f(x)$ does not align with the units for the domain of $g(x)$.

**Activity 5 - Function Composition with Words**

We’ve developed an understanding now of where function composition can go wrong with the inputs and outputs not aligning symbolically, numerically, and with units. In this last activity, we will deepen our appreciation for the importance of units and context to see where function composition can be helpful or fails to be useful.

**Part I-** The function $D(p)$ gives the number of items that will be demanded (by the consumers/buyers) when the price is $p$ in dollars. The production cost $C(x)$ is the cost (in dollars) of producing $x$ items. To determine the cost of production when the price is $6, you would do which of the following, and why?

* 1. Evaluate D(C(6))
	2. Evaluate C(D(6))
	3. Solve D(C(x)) = 6
	4. Solve C(D(x)) = 6

The answer is b, answers may vary. E.g. If the price is $6, D(6) tells the demand (in number of items), so C(D(6)) tells us the production costs in dollars.

**Part II-** The function $C(k)$ gives the energy bill in dollars for $k$ kilowatt-hours (KWHr) of charging, and the function $E(d)$ gives energy use (in KWHr) for $d$ miles of travel in one’s electric vehicle.

1. Which function composition order makes sense? $(C ∘E)(d)$
2. What are the input units for your function composition? miles
3. What are the output units for your function composition? dollars
4. In your own words, what does this composite function tell you? Answers may vary slightly, e.g. this function tells you the amount it costs you to charge your car from driving d miles

**Part III-** Your town’s waste treatment plant operator has developed two functions. The first function $L(x),$ determines how much solution of a cleaning agent to use in liters based upon the number of millions of gallons of wastewater that have come into the plant. The second function $G(x)$ gives the amount of money in dollars needing to be budgeted based on the number of liters of solution used.

1. Which function composition makes sense in this situation? $(G∘L)(x)$ or $(L∘G)(x)$? $(G∘L)(x)$
	1. What are the input and output units of the appropriate composition? x million gallons for the input and $G(L(x))$ dollars for the output
	2. In your own words, what does this composite function tell you? Answers may vary slightly, e.g. this function tells you the amount of money to budget for the cleaning agent based upon the number of million gallons of waste to be treated.
2. Suppose the treatment plant operator is looking at the meter’s readings for the past week and has the following facts and figures (not necessarily in order of how they were computed):
* 17.5 million gallons of wastewater came through the plant
* $2,132.31 is to be allocated for the cleaning agent
* 1,008 Liters of cleaning agent were used

 Which of the following pairs of function values make sense?

1. $L(1,008) = 17.5 ; G(17.5) =2,132.31$
2. $L(17.5) = 1,008 ; G(1,008) = 2,132.31$
3. $L(1,008) = 2,132.31; G(2132.31) = 17.5$
4. $L(17.5) =1,008; G(1,008) = 2,132.31$

D