# Dice and Distributions

In this activity, we'll use spreadsheets to take an experimental approach to expectation and distributions that arise from scenarios that involve the rolling of dice. We will assume all dice are fair for this activity.

## Part I: Single Die

In this section, we'll take a look at what happens when we repeatedly roll a single die.

1. What are the possible outcomes of rolling a single 6 sided die?

   1,2,3,4,5,6

2. What is the probability of rolling a die and getting a 4? a 5?

   Both are equally likely to occur, with probability 1/6.

3. If you were to roll a 6 sided die 600 times, how many of those times would you predict that the result is 4?

   Since $P(\text{rolling a } 4) = \frac{1}{6}$, we'd expect to get 4 as a result about $600 \cdot \frac{1}{6} = 100$ times.

4. What is the expected value of rolling a fair 6 sided die?

   Since each outcome has probability $\frac{1}{6}$ of occurring, we have that

$$\text{expected value} = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5$$

5. How would you interpret your last answer?

   If you repeatedly roll the die, you'd expect that the average outcome is about 3.5 (even though it is impossible to roll a 3.5).

6. Use the provided Excel sheet titled 'many toss simulator' to simulate 10 single die rolls. You can do this by selecting the first sheet (called single die roll) at the bottom left of the page and select the cell A3. Type '=RANDBETWEEN(1,6)' without quotation marks to simulate rolling a die. Select the lower right corner of cell A3 and drag this to 9 cells below (to A12) to copy this formula to the selected cells. This simulates 10 independent rolls of a single die.

   (a) What is the average result of these 10 rolls? (Select the 10 cells with random values- the average is displayed at the bottom of the window.) Does this agree with your calculation in problem 4? Why or why not?

   Answers will vary. But it could look something like: the average is 3.3, which is smaller than the expected value we computed earlier. This should get closer if we roll the die more times.

   (b) Use the provided 'Frequency of Value' table to create a histogram of your results. To do this, highlight the values in the frequency table (cells E2 to J3), and select the 'insert' tab at the top. Then click 'recommended charts', and scroll down and select 'clustered column.' Add the title 'Frequency of Values' at the top of your histogram. (Optional) You can format this table to your liking.

   (c) Hit Shift+F9 to recalculate your worksheet (this is like rolling a die 10 times again). What is the average outcome this time? Did your answer change from the last time you did this? Did the histogram change? Can you explain why?

   Answers will vary. It could look like: The average is 4.7 this time, much larger than before. Both the average and the histogram changed. This is because we got different values when we re-rolled the die 10 times.

   (d) Make a prediction about what happens to the average outcome if we were to change the number of rolls from 10 to 400.

   We would predict that after more rolls, the average value should be closer to the expected value we computed in problem 4.

   (e) Use the Excel spreadsheet to simulate 400 die rolls and compute the average value (select the bottom right of cell A12, and drag down to cell A402. Read the average value from the bottom of the window). How does this compare to the expected value you computed in 4?

   Answers will vary. It could look like: The average this time is about 3.62, which is close to the expected value of 3.5.

## Part II: Two Dice

Suppose you roll two fair six sided dice, and record the resulting sum.

1. What are the possible values for rolling a pair of dice?

   2,3,4,5,6,7,8,9,10,11,12

2. What is the probability of rolling a 4? a 5? Why are these answers different from those by rolling a single die? (If you are stuck, try moving to the next three questions, but make sure to come back!)

   Since there are 6 sides to each die, between the pair there are 36 ways the pair of dice may land (this follows from multiplication rule for counting. Distinguish the dice, by calling one die A, and the other die B. A four may be obtained if both dice result in a 2, or if die A is 1 while die B is 3, or vice versa. There are 3 ways to get a 4 from rolling a pair of dice, so the probability of that occurring is $\frac{3}{36}$. Similar reasoning may be used to find the probability of obtaining a 5 is $\frac{4}{36}$.

   Alternatively, students may jump to 5, and refer to the table to come up with their answers.

3. Which value do you wager has the highest probability of occurring when rolling a pair of dice? Explain your reasoning.

   The value that is most likely to occur is 7, since this outcome can happen by rolling a (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) (students may use their table in problem 5 to arrive at this.)

4. Can you hypothesize what the expected value of tossing a pair of dice is? Justify your hypothesis.'

   Answers will vary. They may reason that 7 is the expected value since it is right in the middle of all the possible outcomes.

5. Make a table that shows all possible outcomes of rolling a pair of dice (you can do this by hand or in Excel).

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

6. Using your table, compute the expected value of rolling a pair of dice? How does this compare to the expected value of rolling a single dice?

$$\text{expected value} = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) = 7$$

This is double what we got for the expected value of rolling a single die. Indeed, note that $2 \cdot 3.5 = 7$

7. Use your observation in the previous problem to make an inference as to the expected value of tossing 3 fair six sided dice.

We would guess that the expected value of rolling 3 dice would be $3 \cdot 3.5 = 10.5$

8. Based on what you've learned about tossing 2 dice, can you write a formula for the expected value of tossing $n$ fair six sided dice? Explain your reasoning.

Since we know that a single die has expected value of 3.5, we'd expect that $n$ of them will result in the expected value of $n \cdot 3.5$.

9. Based on the table you made, what would you predict a histogram depicting many tosses of a pair of dice to look like?

Since 7 is the most likely value to occur, we'd surmise that the histogram will peak at 7. Since 2 and 12 have the lowest likelihood of occurring, we'd further posit that the distribution tapers off on both ends.

10. Use Excel to simulate rolling a pair of dice 400 times. To do this, navigate to the second sheet (on the bottom left of the window) titled, 'Pair of Dice.' Select cell A3 and type '=RANDBETWEEN(1,6) + RANDBETWEEN(1,6)' to properly simulate tossing a pair of dice.

   a) What is the average result of these 400 tosses? (Drag the bottom right corner of cell A3 to cell A402. Read the average value from the bottom of the window.) Was this result surprising? Explain why or why not.
   Answers will vary. It could look like: The average value is about 7.04. This was not surprising at all because we knew the expected value of tossing a pair of dice would be 7. Since we simulated rolling a pair of dice many times, the average should be close to that exact expected value.

   b) Use the provided frequency table to make a histogram showing the frequency of outcomes. (To do this, follow the same instructions per Part I problem 6b.) Explain why this looks different from rolling a single die.

   This looks a mound, with the peak occurring for outcome of 7. This is because each outcome has a different number of ways of being obtained, as there are two dice at play.

### Part III: a Simple Game

In this part, you will play a simple dice game (with pretend monetary stakes) and investigate the expected payout of the game. The game is a single player game (though your group may pretend to be one person) and is played as follows: you roll a fair pair of 6 sided dice (or use the Excel template titled 'Single toss simulator' to simulate rolling a pair of dice) and record the total,

i) if the total is 2, you win $\$2$

ii) if the total is 7, you win $\$.50$

iii) if the total is 11 or 12, you must pay $\$1$

iv) if the total is 5 or 9, then you must pay $\$.25$

v) if the total is 3,4,6,8, or 10, you must pay $\$.10$

1. Based on these rules, posit whether you are more likely to win money or to lose money. Explain yourself.

   Note that the probability of winning in this game is $\frac{1}{36} + \frac{6}{36} = \frac{7}{36} \approx .19$ while the probability of losing money is $\frac{5+5}{36} + \frac{1}{36} = \frac{11}{36} \approx .31$, so you are more likely to lose money, than to win.

2. Play at least 10 rounds of the game. For each round, record the outcome of the toss, as well as the monetary outcome of each round. If you win money, denote the payment as positive (as you will be receiving money) and if you must pay, denote the payment as a negative value (as you will be losing money). You may record your results in an Excel table or in the space below.

   Answers will vary. It may look like:

| Outcome | Payout |
|---|---|
| 7 | $.50 |
| 8 | -$.70 |
| 5 | $ 0 |
| 6 | -$.70 |
| 3 | $ 0 |
| 2 | $ 2 |
| 6 | -$ .70 |
| 9 | $ 0 |
| 6 | -$.70 |
| 6 | -$.70 |

3. After all 10 rounds of the game, did you end up making money, or losing money? And how much?

   Answer will depend on 2. Here, we have $\$.50 - \$.70 + \$0 - \$.70 + \$0 + \$2 - \$.70 + \$0 - \$.70 - .\$.70 = -\$1$

4. What is the expected payout (the amount of money gained or lost) if you play this game just one time? Knowing the expected value, would you play this game? Why or why not?

   There are four possible outcomes with payment per the rules, so we have

$$\text{expected value} = \$2\left(\frac{1}{36}\right) + \$.40\left(\frac{6}{36}\right) - \$.70\left(\frac{5+5}{36}\right) - \$1\left(\frac{1}{36}\right) = -\$.10$$

   We ought not play this game since the expected outcome is negative, that means we are expected to lose money!

5. Use the expected payout you calculated to make a prediction on how much money you would make, or lose, after 10 rounds of playing this game. How does this compare to your actual outcome of playing 10 rounds?

   We would expect after 10 rounds, to make $10 \cdot (-\$.10) = -\$1$. In other words, we expect to lose $\$1$ after playing 10 rounds of this game.

   Answers will differ in comparison. Here, we lost exactly what we were expected to lose- which was $\$1$ after playing 10 rounds of the game.

6. Write a formula that describes the expected payout after $n$ rounds of playing this game.

   Since the expected value of this game is $-\$.10$, after $n$ rounds, we would expect to make

$$n(-\$.10) = -\$\frac{n}{10}.$$

   That is, we will expect to lose $\$\frac{n}{10}$ after $n$ rounds.