Functions and Modeling ARC on Function Transformations

COVER PAGE

Transformations of Functions ARC

Designated Course
Functions and Modeling

Targeted Topics
Function Transformations

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Description of the ARC

Short Abstract

In the Functions and Modeling class, there is a high emphasis on establishing the relationship between the algebraic and the graphical representations of functions. One way to underline this is through the study of function transformations. The goal of these activities is to provide a more hands-on, experimentation-oriented component that will allow students to more easily generalize the effects of a set of function transformations on a graph. In doing so, students should have concrete, visual examples to help solidify their understanding of the algebraic and graphical relationship between functions.

Long Abstract

In the Functions and Modeling class, there is a high emphasis on establishing the relationship between the algebraic and the graphical representations of functions. One such way to establish this relationship that also challenges students to think about how two quantities can change simultaneously is through the study of function transformations. The goal of these activities is to take some already scaffolded exercises that explore the concepts of function transformations and provide a more hands-on, experimentation-oriented component that will allow students to more easily generalize the effects of a set of function transformations on a graph. In doing so, students should have concrete, visual examples to help solidify their understanding of the algebraic and graphical relationship between functions. Additionally, this activity should strengthen their covariational reasoning skills as they move from observing two quantities moving simultaneously to predicting how the two quantities will move simultaneously.

Instructor Notes for all Activities

The main ideas these activities are targeting are transformation effects on:

1. Individual points
2. The graph as a whole
3. Domain and range

Throughout the activities, instructors should look for opportunities to emphasize the values of the sliders, how they affect the direction of the transformation, and *why* they affect the transformation like they do. Activity 3 especially emphasizes the order of the transformations when there are more than one of the same kind. In all activities, the authors have found it best for the *why* of transformations to be discussed as a class, rather than realized through an isolated activity. Therefore, it is not recommended that instructors rely on the activities to fully explain why transformations operate as they do; rather the activities should be incorporated into a classroom discussion.

This activity works under the assumption that:

1. Instructors have already introduced the language and syntax necessary to do function transformation exercises.
2. Instructors have already provided an example of all the transformations one-by-one (in our case with a geogebra applet already in existence).

Note: The tables in Desmos should not be altered by students in Activity 1, but will be altered by students in the optional extension in Activity 2.

Activity 1 - Vertical Transformations

Description:
This activity allows students to experiment within Desmos to explore the relationships between algebraic changes and the resulting graphical transformations. Activity 1 focuses on basic transformations where students explore one vertical transformation type at a time. Additionally, students should be able to relate the algebraic expression of the function transformation and the change in the original function's domain and range.

Activity Instructions:
https://www.desmos.com/calculator/wkhmtawkzp
In Desmos (click here or copy the url above), you have a parent graph of $f(x) = \sqrt{x}$ graphed with a domain of [0,16] and a range of [0, 4]. The sliders on the left help create a transformation graph, $g(x) = af(bx - c) + d$ where in this case $b = 1$ and $c = 0$. Move the sliders to answer the following questions.

(a) Which slider will shift the graph up and down? Which slider will vertically stretch or compress the graph?

(b) For each of the following cases, find the equation for $g(x)$ and the range for $g(x)$.
   (i) Vertically shift the graph up 3 units.
      (1) Using the graphs of $f(x)$ and $g(x)$ and/or the tables of $f(x)$ and $g(x)$, describe what happened to the x-values during the transformation. Similarly, describe what happened to the y-values. Use the placement of the 3 in your equation to explain your response.
   (ii) Vertically shift the graph down 2 units.
      (1) Using the graphs of $f(x)$ and $g(x)$ and/or the tables of $f(x)$ and $g(x)$, describe what happened to the x-values during the transformation. Similarly, describe what happened to the y-values. Use the placement of the 2 in your equation to explain your response.
   (iii) Vertically stretch the graph by a factor of 3.
      (1) Using the graphs of $f(x)$ and $g(x)$ and/or the tables of $f(x)$ and $g(x)$, describe what happened to the x-values during the transformation. Similarly, describe what happened to the y-values. Use the placement of the 3 in your equation to explain your response.
   (iv) Vertically compress the graph by a factor of 2.
      (1) Using the graphs of $f(x)$ and $g(x)$ and/or the tables of $f(x)$ and $g(x)$, describe what happened to the x-values during the transformation. Similarly, describe what happened to the y-values. Use the placement of the 2 in your equation to explain your response.

(c) What change in the domain of $f(x)$ occurred in the above transformations? Again use the placement of the **a** and the **d** in your equations to explain your response.

(d) Suppose $d = 0$. What value of **a** would make the range of $g(x)$ equal to $[0, 24]$?

(e) Suppose $d = 0$. What value of **a** would make the range of $g(x)$ equal to $[-\ 8, 0]$?
(f) Suppose $a = 1$. What value of **d** would make the range of $g(x)$ equal to $[7, 11]$?

Activity 2 - Horizontal Transformations

Description:

This activity allows students to experiment within Desmos to explore the relationships between algebraic changes and the resulting graphical transformations. Activity 2 focuses on basic transformations where students explore one horizontal transformation type at a time. Additionally, students are prompted to explore why horizontal transformations move in one direction when the notation might suggest the opposite direction.

Activity Instructions:

https://www.desmos.com/calculator/6lta79xbov

In Desmos (click here or copy the url above), you have a parent graph of $f(x) = \sqrt{x}$ graphed with a domain of [0,16] and a range of [0, 4]. The sliders on the left help create a transformation graph, $g(x) = af(bx - c) + d$ where in this case $a = 1$ and $d = 0$. Move the sliders to answer the following questions.

(a) Which slider will shift the graph left and right? Which slider will horizontally stretch or compress the graph?

(b) For each of the following cases, find the equation for $g(x)$ and the domain for $g(x)$.
   - (i) Horizontally shift the graph right 3 units.
   - (ii) Horizontally shift the graph left 2 units.
   - (iii) Horizontally stretch the graph by a factor of 2.
   - (iv) Horizontally compress the graph by a factor of 3.

(c) Suppose $c = 0$. What value of **b** would make the domain of $g(x)$ equal to $[0, 32]$?

(d) Suppose $b = 1$. What value of **c** would make the domain of $g(x)$ equal to $[-7, 9]$?

(e) From the above exercises, you can observe that the arithmetic for the transformations seems backwards (especially when compared with the vertical transformations). In this exercise, we are going to explore *why* this is the case.
   - (i) Consider $g(x) = f(x - 5)$. Write out what $g(2)$ is.
      - i. Notice that 2 is the input of $g(x)$, not $f(x)$. What is the input of $f(x)$ when 2 is the input of $g(x)$?
         - *Note: here, we have transformed backwards (we went from an input of $g(x)$ to an input of $f(x)$).*
      - ii. Now let 2 be an input of $f(x)$. What is the input of $g(x)$ when 2 is the input of $f(x)$?
         - *Note: here, we have transformed in the usual direction (we went from an input of $f(x)$ to an input of $g(x)$).*
      - iii. What is the input of $g(x)$ when 3 is the input of $f(x)$? What is the input of $g(x)$ when 4 is the input of $f(x)$? In general, what is the input of $g(x)$ when x_0 is *any* input of $f(x)$?
         - *We should now have an idea as to why the arithmetic for the transformations seems backwards. The answer lies in who the x-value being transformed belongs to (i.e. which function's domain*

is it in?), and who the x-value in the algebraic expression of the function notation belongs to.

Optional Extension for i-iv in part b above:

- Using the graphs of $f(x)$ and $g(x)$, and the labeling in Desmos, write down the points for R, P, and S on $f(x)$ and the points RT, PT, and ST on $g(x)$. Using these points, describe what happened to the x-values during the transformation. Similarly, describe what happened to the y-values. Use the placement of the 3 (or 2) in your equation to explain your response.
    - Now, you control the x-values in the table for $g(x)$. Find what x-values give you a table for $g(x)$ that has the same y-values as the $f(x)$ table. Compare the x-values in the $g(x)$ table with the x-values in the $f(x)$ table. What mathematical operations should be applied to the x-values in the $f(x)$ table to obtain the x-values in the $g(x)$ table?

Activity 3 - Transformation Orders

Description:

This activity allows students to experiment within Desmos to explore the relationships between algebraic changes and the resulting graphical transformations. Activity 3 focuses on transformations where students explore two transformation types at a time. First, the order of vertical transformations is considered, then the order of horizontal transformations.

Activity Instructions:

https://www.desmos.com/calculator/frwmusfw1v

In Desmos (click here or copy the url above), you have a parent graph of $f(x) = \sqrt{x}$ graphed with a domain of [0,16] and a range of [0, 4]. The sliders on the left help create a transformation graph, $g(x) = af(bx - c) + d$. Move the sliders to answer the following questions.

(a) This question will be seeking to help you understand how the **order** of the vertical transformations matter by seeing where the point P goes.

- (i) Let **d = 1**. What is the new point PT? Now, keep **d = 1** and let **a = 3**. What is the new point PT?
- (ii) Reset your sliders (**a = 1** and **d = 0**). Now let **a = 3**. What is the new point PT? Now, keep **a = 3** and let **d = 1**. What is the new point PT?
- (iii) In the previous two parts, PT should be the same point since **a = 3** and **d = 1** give us the same g(x). However, it is NOT true that we can apply transformations in any order we want. Desmos is accounting for the order for us. Since P has a y-value of 2 and our PT ends up with the y-value of of 7, which of the following gets us to 7 when we algebraically try to find PT?
  - (1) $(2 + 1) \cdot 3$
  - (2) $3 \cdot 2 + 1$
- (iv) Which of the following corresponds with the arithmetic you chose?
  - (1) Applying **a = 3** and then **d = 1**
  - (2) Applying **d = 1** and then **a = 3**
- (v) What order should be used when applying two vertical transformations?

(b) This question will be seeking to help you understand how the **order** of the horizontal transformations matter by seeing where the point P goes. If you haven't already, reset your sliders (**a = 1** and **d = 0**).

- (i) Let **b = 3**. What is the new point PT? Now, keep **b = 3** and let **c = 2**. What is the new point PT?
- (ii) Reset your sliders (**b = 1** and **c = 0**). Now let **c = 2**. What is the new point PT? Now, keep **c = 2** and let **b = 3**. What is the new point PT?
- (iii) In the previous two parts, PT should be the same point since **c = 2** and **b = 3** give us the same g(x). However, it is NOT true that we can apply transformations in any order we want. Desmos is accounting for the order for us. Since P has a x-value of 4 and our PT ends up with the x-value of of 2, which of the following gets us to 2 when we algebraically try to find PT?

(1) $\frac{1}{3} \cdot 4 + 2$

(2) $(4 + 2) \cdot \frac{1}{3}$

(iv) Which of the following corresponds with the arithmetic you chose?

(1) Applying **b = 3** and then **c = 2**

(2) Applying **c = 2** and then **b = 3**

(v) What order should be used when applying two horizontal transformations?

(c) What values of **a** and **d** would make the range of $g(x)$ equal to $[-2, 10]$?

(d) What values of **b** and **c** would make the domain of $g(x)$ equal to $[-18, 14]$?

(e) Describe how the slider for **a** affects the graph of $g(x)$. What do you notice about the relationship between the value of **a** and the severity of the transformation?

(f) Repeat the previous questions with the sliders for **b**, **c**, and **d**.