Avoid Memorization: Deriving Identities

Introduction: Overwhelmed with needing to know numerous trigonometric identities? This activity will demonstrate that you only need to memorize a few identities and others will follow. Let's explore how you can build additional identities using what you already know and hence free up some space in your brain.

Prerequisites: The building blocks that you will be using to create new identities include the reciprocal and quotient identities, as well as the following.

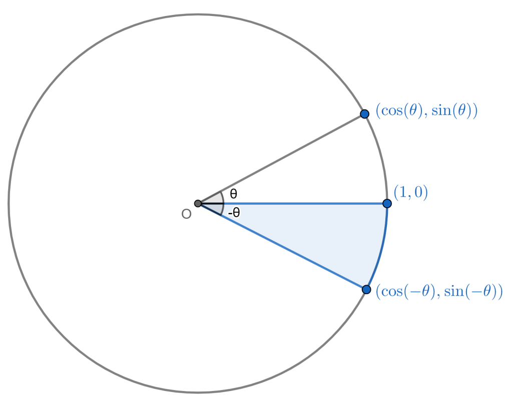
$\sin^2\theta + \cos^2\theta = 1$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

Recall: You will also be using your prior knowledge of even and odd functions.

- The sine function is an odd function, so $\sin(-\theta) =$ ___$-\sin\theta$___.
- The cosine function is an even function, so $\cos(-\theta) =$ ___$\cos\theta$___.



Let's get warmed up! Try this! Determine whether the tangent function is an even or odd function and complete the sentence below. Justify your answer.

- The tangent function is an ___odd___ function, so $\tan(-\theta) =$ ___$-\tan\theta$___.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Part 1: Creating the Pythagorean Identities

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{1}$$
$$\tan^2\theta + 1 = \sec^2\theta \tag{2}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{3}$$

The primary building block in this section is the first Pythagorean Identity in the above list. The goal of this section is to create the other two Pythagorean Identities.

a) Read through the explanation on how we can create Identity (2) by using Identity (1).

Begin with Identity (1) and divide each term by $\cos^2\theta$.

$$\sin^2\theta + \cos^2\theta = 1 \qquad (1)$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Simplify each term using the reciprocal and quotient identities.

$$\tan^2\theta + 1 = \sec^2\theta$$

This is Identity (2)!

b) **Now it's your turn!** Begin with Identity (1) and create Identity (3) in a similar fashion.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Longrightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Longrightarrow 1 + \cot^2\theta = \csc^2\theta$$

Part 2: Creating the Double-Angle Identities for Cosine

The primary building block we are going to use for this section is:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \tag{4}$$

Our goal is to create the following identities.

$$\begin{align} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \tag{5} \\ &= 2\cos^2\alpha - 1 \tag{6} \\ &= 1 - 2\sin^2\alpha \tag{7} \end{align}$$

Oftentimes in mathematics, we need to use a clever substitution in order to apply a formula or identity.

a) Read through the explanation of how to create Identity (5) by using Identity (4) with a clever substitution.

   Begin with Identity (4) and substitute β with α.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (4)$$

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$

   Simplify the equation.

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$$

   This is Identity (5)!

b) **It's your turn (with some guidance)!** Create Identity (6) by completing the steps below. That is, derive $\cos(2\alpha) = 2\cos^2\alpha - 1$.

Step 1: Consider Identities (1) and (5). Write these equations below using α as the argument in both equations and label them with the Identity number for reference.

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{1}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \tag{5}$$

Step 2: The identity that we are trying to obtain (Identity (6)) only involves the cosine function. So, we need a substitution for $\sin^2\alpha$ in Identity (5) that is in terms of cosine. To do this, use the Pythagorean Identity (1) in Step 1 to write $\sin^2\alpha$ in terms of cosine. Complete the equation below with your result.

$\sin^2\alpha =$ $\underline{\quad 1 - \cos^2\alpha \quad}$

Step 3: Substitute the equivalent expression you found for $\sin^2\alpha$ into Equation (5) and simplify.

$$\begin{aligned} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= \cos^2\alpha - (1 - \cos^2\alpha) \\ &= \cos^2\alpha - 1 + \cos^2\alpha \\ &= 2\cos^2\alpha - 1 \end{aligned}$$

Congratulations! You derived Identity (6)!

c) **Now create Identity** (7) **using a similar method as above.**

$$\begin{aligned} &\sin^2\alpha + \cos^2\alpha = 1 \qquad (1) \\ \Longrightarrow\ &\cos^2\alpha = 1 - \sin^2\alpha \end{aligned}$$

$$\begin{aligned} \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= (1 - \sin^2\alpha) - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha \end{aligned}$$

You're becoming an expert with substitution!

Part 3: Creating the Double-Angle Identity for Sine

The primary building block we are going to use for this section is:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{8}$$

Your goal: Create the following identity.

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \tag{9}$$

To obtain Identity (9), determine an appropriate substitution for β in Identity (8) and simplify.

Hint: If you need assistance, refer to the beginning of Part 2 where we derived Identity (5), a double-angle identity for cosine.

Consider Identity (8), which is $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Substitute in α for β. We then obtain:

$$\begin{aligned}\sin(\alpha+\alpha) &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha\\ &= \sin\alpha\cos\alpha + \sin\alpha\cos\alpha\\ &= 2\sin\alpha\cos\alpha.\end{aligned}$$

Therefore, $\sin(2\alpha) = 2\sin\alpha\cos\alpha$.

Part 4: Creating the Half-Angle Identities for Cosine and Sine

The building blocks we are going to use for this section are identities that we have already derived. More specifically, we will be utilizing Identities (6) and (7), which are listed below for reference.

$$\cos(2\alpha) = 2\cos^2\alpha - 1 \qquad (6)$$

$$\cos(2\alpha) = 1 - \sin^2(\alpha) \qquad (7)$$

The goal for this section is to create the half-angle identities for sine and cosine.

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad (10)$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad (11)$$

We will begin by showing the details of how to create Identity (10). After you understand the idea behind the derivation, you will be asked to create Identity (11).

a) Read through the explanation on how we can create Identity (10).

Begin with Identity (6), which is $\cos(2\alpha) = 2\cos^2\alpha - 1$.

Substitute $\frac{\alpha}{2}$ in for α. We then obtain:

$$\cos\left(2\left(\frac{\alpha}{2}\right)\right) = 2\cos^2\left(\frac{\alpha}{2}\right) - 1.$$

Simplify the equation.

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1$$

Solve for $\cos\left(\frac{\alpha}{2}\right)$.

$$\begin{aligned} \cos\alpha &= 2\cos^2\left(\frac{\alpha}{2}\right) - 1 \\ \Longrightarrow\ 2\cos^2\left(\frac{\alpha}{2}\right) &= 1+\cos\alpha \\ \Longrightarrow\ \cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\alpha}{2} \\ \Longrightarrow\ \cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \end{aligned}$$

b) **It's Your Turn!** Create Identity (11) using an appropriate substitution in (7).

Begin with Identity (7), which is $\cos(2\alpha) = 1 - 2\sin^2\alpha$.

Substitute $\frac{\alpha}{2}$ in for α. We then obtain:

$$\cos\left(2\left(\frac{\alpha}{2}\right)\right) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right).$$

Simplify the equation.

$$\cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

Solve for $\sin\left(\frac{\alpha}{2}\right)$.

$$\begin{aligned} \cos\alpha &= 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \\ \implies 2\sin^2\left(\frac{\alpha}{2}\right) &= 1 - \cos\alpha \\ \implies \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\cos\alpha}{2} \\ \implies \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \end{aligned}$$

Part 5: Creating the Difference Identities for Sine and Cosine

The building blocks in this section are the Sum Identities for Cosine and Sine. These are Identities (4) and (8), respectively. They are listed below for your reference.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (4)$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (8)$$

The goal of this section is to create the Difference Identities for Cosine and Sine. They are as follows.

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (12)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (13)$$

a) Determine an appropriate substitution for β in Identity (4) to create Identity (12).

Hint: You will need to use your knowledge regarding even and odd functions when you simplify.

Consider Identity (4), which is $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Substitute in $-\beta$ for β. We then obtain:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Therefore, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

b) Use a similar method to create Identity (13) using Identity (8).

Consider Identity (8), which is $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

Substitute in $-\beta$ for β. We then obtain:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha(-\sin\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta.\end{aligned}$$

Therefore, $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Part 6: Verifying the Product-to-Sum Identities

In this final section, we will focus on verifying the Product-to-Sum Identities using problem-solving strategies such as multiplying by a clever form of "1" and adding a clever form of "0".

The building blocks in this section are the Sum and Difference Identities for Cosine and Sine. They are listed below with the Identity reference number used previously in this activity.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (4)$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (8)$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (12)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \qquad (13)$$

The Product-to-Sum Identities we will be verifying are the following.

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right] \qquad (14)$$
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right) - \cos\left(\alpha+\beta\right)\right] \qquad (15)$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right] \qquad (16)$$

a) Read through the step-by-step verification of Identity (14), which is rewritten below with its reference number.

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right] \qquad (14)$$

Preview:

- We will be using a common problem-solving strategy of rewriting the integer "1" in a clever form. Since multiplying an expression by "1" creates an equivalent expression, it is sometimes useful to rewrite "1" in a clever way.
- We will also be using the problem-solving strategy of rewriting the integer "0" in a clever form. Since adding "0" to an expression creates an equivalent expression, it is sometimes useful to rewrite "0" in a clever way.
- We will be using Identities (4) and (12). They are included below for reference (along with their Identity reference number from earlier in the activity).

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad (4)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (12)$$

Observations:

- Each identity has a $\cos\alpha\cos\beta$ term.
- Each identity has a $\sin\alpha\sin\beta$ term; although, the terms have opposite signs.
- We will be creative and use the fact that "$-\sin\alpha\sin\beta+\sin\alpha\sin\beta=0$."

Goal: Verify Identity (14): $\cos\alpha\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right].$

Here we go!

$$
\begin{aligned}
\cos\alpha\cos\beta &= 1\left(\cos\alpha\cos\beta\right)\\
&= \tfrac{2}{2}\left(\cos\alpha\cos\beta\right)\\
&= \tfrac{1}{2}\left(2\cos\alpha\cos\beta\right)\\
&= \tfrac{1}{2}\left[\cos\alpha\cos\beta+\cos\alpha\cos\beta\right]\\
&= \tfrac{1}{2}\left[\cos\alpha\cos\beta+0+\cos\alpha\cos\beta\right]\\
&= \tfrac{1}{2}\left[\cos\alpha\cos\beta+\left(-\sin\alpha\sin\beta+\sin\alpha\sin\beta\right)+\cos\alpha\cos\beta\right]\\
&= \tfrac{1}{2}\left[\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sin\alpha\sin\beta+\cos\alpha\cos\beta\right]\\
&= \tfrac{1}{2}\left[\left(\cos\alpha\cos\beta-\sin\alpha\sin\beta\right)+\left(\sin\alpha\sin\beta+\cos\alpha\cos\beta\right)\right]\\
&= \tfrac{1}{2}\left[\left(\cos\alpha\cos\beta-\sin\alpha\sin\beta\right)+\left(\cos\alpha\cos\beta+\sin\alpha\sin\beta\right)\right]\\
&= \tfrac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]
\end{aligned}
$$

Teacher Note: These identities can also be verified by starting with the right-hand side and substituting in the appropriate Sum and Difference Identities for Sine and Cosine. Even though the method in which we wrote up the solution involves more thought, it demonstrates a very useful problem-solving skill of creatively adding "0" and multiplying by "1."

b) **Now it's your turn (with some guidance)**

Goal: Verify Identity (15): $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$.

Fill in the blanks with an appropriate mathematical expression.

$$\begin{aligned}
\sin\alpha\sin\beta &= 1(\sin\alpha\sin\beta) \\
&= \tfrac{2}{2}(\sin\alpha\sin\beta) \\
&= \tfrac{1}{2}\left(\boxed{2\sin\alpha\sin\beta}\right) \\
&= \tfrac{1}{2}\left[\sin\alpha\sin\beta + \boxed{\sin\alpha\sin\beta}\right] \\
&= \tfrac{1}{2}[\sin\alpha\sin\beta + 0 + \sin\alpha\sin\beta] \\
&= \tfrac{1}{2}\left[\sin\alpha\sin\beta + \left(\boxed{\cos\alpha\cos\beta} - \boxed{\cos\alpha\cos\beta}\right) + \sin\alpha\sin\beta\right] \\
&= \tfrac{1}{2}\left[\sin\alpha\sin\beta + \boxed{\cos\alpha\cos\beta} - \boxed{\cos\alpha\cos\beta} + \sin\alpha\sin\beta\right] \\
&= \tfrac{1}{2}\left[(\sin\alpha\sin\beta + \cos\alpha\cos\beta) - \left(\boxed{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\right)\right] \\
&= \tfrac{1}{2}\left[\cos(\alpha-\beta) - \boxed{\cos(\alpha+\beta)}\right]
\end{aligned}$$

Teacher Note: After an appropriate amount of time, if a student is stuck, consider suggesting that they work backwards. Working backwards is another useful problem-solving tool for students to add to their toolbelt.

c) **Try this last one!**

Goal: Verify Identity (16): $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$.

$$\begin{aligned}
\sin\alpha\cos\beta &= 1\left(\sin\alpha\cos\beta\right) \\
&= \tfrac{2}{2}\left(\sin\alpha\cos\beta\right) \\
&= \tfrac{1}{2}\left(2\sin\alpha\cos\beta\right) \\
&= \tfrac{1}{2}\left[\sin\alpha\cos\beta+\sin\alpha\cos\beta\right] \\
&= \tfrac{1}{2}\left[\sin\alpha\cos\beta+0+\sin\alpha\cos\beta\right] \\
&= \tfrac{1}{2}\left[\sin\alpha\cos\beta+(\sin\beta\cos\alpha-\sin\beta\cos\alpha)+\sin\alpha\cos\beta\right] \\
&= \tfrac{1}{2}\left[\sin\alpha\cos\beta+\sin\beta\cos\alpha-\sin\beta\cos\alpha+\sin\alpha\cos\beta\right] \\
&= \tfrac{1}{2}\left[(\sin\alpha\cos\beta+\sin\beta\cos\alpha)+(\sin\alpha\cos\beta-\sin\beta\cos\alpha)\right] \\
&= \tfrac{1}{2}\left[(\sin\alpha\cos\beta+\sin\beta\cos\alpha)+(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\right] \\
&= \tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]
\end{aligned}$$