Angle Sum Identities

In this activity, we will derive the angle sum identity for cosine:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

This identity (along with the angle sum identity for *sine*) will be extremely important for discovering *new* trig identities.

Part 1: GeoGebra Introduction (guided by your instructor)

GeoGebra Basics (guided by your instructor):

- Open GeoGebra and explore the basic tools for drawing lines, line segments, and angles.
- Practice constructing a triangle and identifying its centroid (or other centers, such as the circumcenter, orthocenter, or incenter)

Exploration Activity:

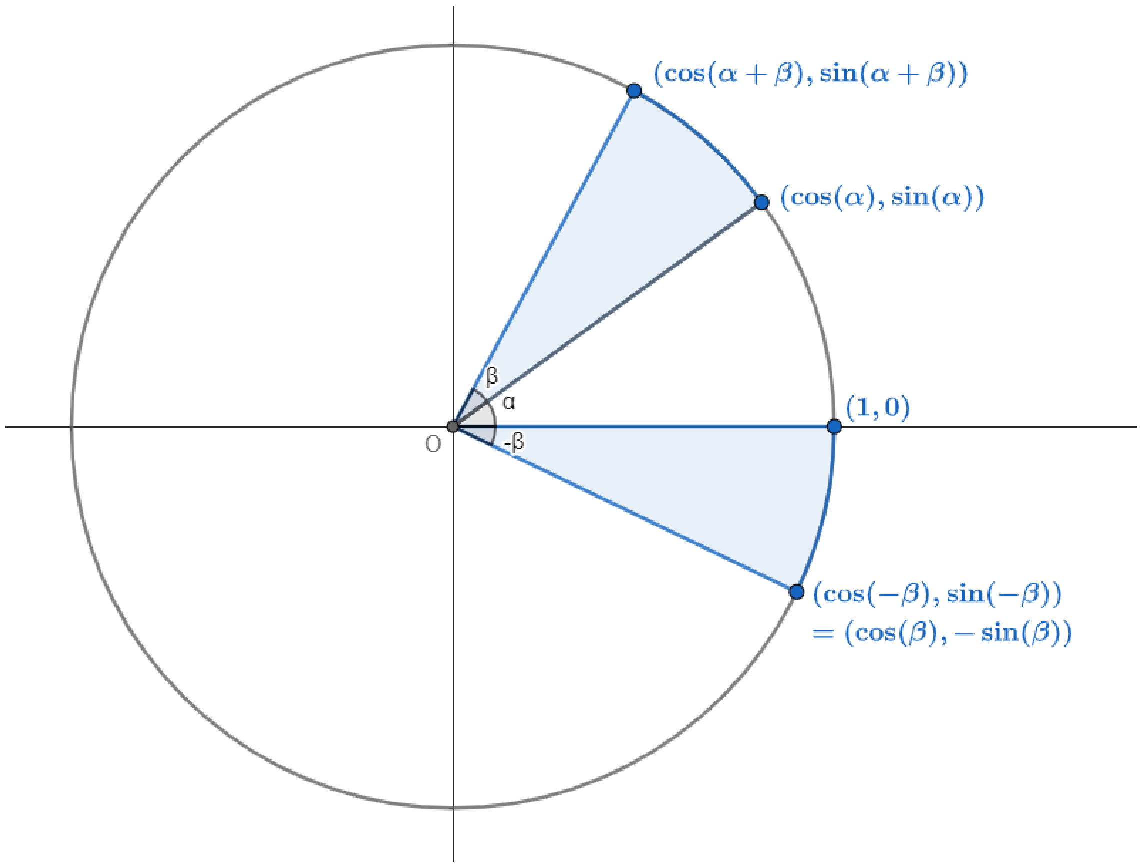
- Watch the one-minute YouTube video explaining why the angles in a triangle sum to 180 degrees.
- Reconstruct the diagrams shown in the video using GeoGebra.

Part 2: Exploring the Unit Circle

- Use GeoGebra to plot the following points in the unit circle (as shown in the diagram on the following page):

  $(1,0)$
  $(\cos(\alpha), \sin(\alpha))$
  $\left(\cos(\alpha+\beta), \sin(\alpha+\beta)\right)$
  $(\cos(-\beta), \sin(-\beta))$

  You may assume that α and β are generic angles, and you should create these points as movable points (except for (1,0) of course).



This diagram can be found here.

- What do you notice about these 4 points? Is there anything special about pairing them up in a certain way (in terms of angles, distances, etc.)?

  Answers will vary. Each of the 4 points is on the unit circle. Students may notice that the angle between $(1,0)$ and $\left(\cos(\alpha+\beta), \sin(\alpha+\beta)\right)$ is the same as the angle between $(\cos(\alpha), \sin(\alpha))$ and $(\cos(-\beta), \sin(-\beta))$. In both cases, the angle is $\alpha+\beta$.

  But more importantly, the distance from $(1,0)$ to $\left(\cos(\alpha+\beta), \sin(\alpha+\beta)\right)$ is the same as the distance from $(\cos(\alpha), \sin(\alpha))$ to $(\cos(-\beta), \sin(-\beta))$.

- Use the distance formula to express the distance between $(1,0)$ and $\left(\cos(\alpha+\beta), \sin(\alpha+\beta)\right)$.

$$\sqrt{(\cos(\alpha+\beta)-1)^2+(\sin(\alpha+\beta)-0)^2}$$

- Use the distance formula to express the distance between $(\cos(\alpha), \sin(\alpha))$ and $(\cos(-\beta), \sin(-\beta))$.

$$\sqrt{(\cos(\alpha) - \cos(-\beta))^2 + (\sin(\alpha) - \sin(-\beta))^2}$$

- What can we say about these two distances? Use your answer to form an identity and simplify completely. Your final answer should give an identity for $\cos(\alpha + \beta)$.

  The distances are equal, so we set:

$$\sqrt{(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2} = \sqrt{(\cos(\alpha) - \cos(-\beta))^2 + (\sin(\alpha) - \sin(-\beta))^2}$$

  We square both sides to eliminate the radicals:

$$(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2 = (\cos(\alpha) - \cos(-\beta))^2 + (\sin(\alpha) - \sin(-\beta))^2$$

  Using that $\cos(-\beta) = \cos(\beta)$ and $\sin(-\beta) = -\sin(\beta)$, we simplify the right-hand side:

$$(\cos(\alpha + \beta) - 1)^2 + (\sin(\alpha + \beta) - 0)^2 = (\cos(\alpha) - \cos(\beta))^2 + (\sin(\alpha) + \sin(\beta))^2$$

  Expanding both sides, we obtain:

$$\cos^2(\alpha + \beta) - 2\cos(\alpha + \beta) + 1 + \sin^2(\alpha + \beta) = \cos^2(\alpha) + \cos^2(\beta) - 2\cos(\alpha)\cos(\beta) + \sin^2(\alpha) + \sin^2(\beta) + 2\sin(\alpha)\sin(\beta)$$

  Using (several times) that $\sin^2\theta + \cos^2\theta = 1$, we simplify to:

$$2 - 2\cos(\alpha + \beta) = 2 - 2\cos(\alpha)\cos(\beta) + 2\sin(\alpha)\sin(\beta)$$

  Subtracting 2 from both sides and then dividing by -2, we obtain the desired identity:

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

- The identity you derived (i.e., $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$) is called the angle sum identity for cosine. Which trig identities did you use in your simplification?

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos(-\beta) = \cos(\beta)$$

$$\sin(-\beta) = -\sin(\beta)$$

- How could this identity be used to find the exact value of $\cos(75°)$? What about $\cos(105°)$? Do this below. [Hint: Is $75°$ the sum of two "nice" angles?]

For $\cos(75°)$, we could let $\alpha = 30°$ and $\beta = 45°$:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(30° + 45°) = \cos(30°)\cos(45°) - \sin(30°)\sin(45°)$$

$$\cos(75°) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$\cos(75°) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

For $\cos(105°)$, we could let $\alpha = 60°$ and $\beta = 45°$:

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(60° + 45°) = \cos(60°)\cos(45°) - \sin(60°)\sin(45°)$$

$$\cos(105°) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$\cos(105°) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

Final answers:

$\cos(75°) = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\cos(105°) = \frac{\sqrt{2}-\sqrt{6}}{4}$

Part 3: Further Exploration

How could we prove the angle sum identity for sine?

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

Could we use a different diagram? Discuss your ideas and draw a possible diagram below.

Answers will vary. Students are encouraged to be creative!

One way to do this is to use the point (0,1) instead of (1,0). Two of the other points can remain the same: $(\cos(\alpha), \sin(\alpha))$ and $(\cos(\alpha + \beta), \sin(\alpha + \beta))$. The fourth point will be $(\cos(90° - \beta), \sin(90° - \beta))$. However, since sine and cosine are complementary "cofunctions," we have

$\cos(90° - \beta) = \sin(\beta)$ and
$\sin(90° - \beta) = \cos(\beta)$.

So, the four points are:

$(0,1)$
$(\cos(\alpha), \sin(\alpha))$
$(\cos(\alpha + \beta), \sin(\alpha + \beta))$
$(\sin(\beta), \cos(\beta))$

After plotting these points, it should be noticed that the distance between the first and third of these points is equal to the distance between the second and the fourth. (The angles between the respective pairs are both $90° - (\alpha + \beta)$.) The angle sum identity for sine can then be derived using the distance formula (in a similar way to how we derived the angle sum identity for cosine).