Angle Sum Identities

Overview

The goal of this activity is for students to derive the angle sum identity for cosine. Students will use GeoGebra (for a geometric visualization), the distance formula, and algebra to understand and prove the identity.

Prerequisite Ideas and Skills

- Students should understand the unit circle definitions of sine and cosine.
- Students should know the distance formula from algebra.
- Students should have basic algebra skills.

Materials Needed to Carry Out the Lesson

- Computers (or tablets) with internet access
- GeoGebra software or access to GeoGebra online
- Activity worksheets
- One-minute YouTube video that explains why the angles in a triangle sum to 180 degrees (Click here to watch the video)

Concepts to be Learned/Applied

- Students will derive the angle sum identity for cosine using a diagram involving the unit circle along with the distance formula. Specifically, students will understand that the distance between the points $(1, 0)$ and $(\cos(\alpha+\beta), \sin(\alpha+\beta))$ is the same as the distance between the points $(\cos(-\beta), \sin(-\beta))$ and $(\cos(\alpha), \sin(\alpha))$ because the points are all on the unit circle and the angle between them in both cases is $\alpha + \beta$. Students will equate these two distances using the distance formula, and will simplify using algebra and the identity $\sin^2 \theta + \cos^2 \theta = 1$ to derive the angle sum identity for cosine: $\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$.
- Students will describe how to similarly derive the angle sum identity for sine.
- Students will learn how to use GeoGebra for geometric constructions and measurements.

Instructional Plan

The lesson begins with an introduction to GeoGebra, where the instructor demonstrates basic functions such as drawing lines and line segments, creating and measuring angles, constructing triangles, and (optionally) identifying different centers of triangles (e.g., centroid, circumcenter, incenter, orthocenter). Note: the instructor should spend some time familiarizing themselves with the Geometry aspects of GeoGebra before beginning this lesson (see the "GeoGebra Instructions: Getting Started" section below). Students will follow along on their devices, replicating the instructor's demonstrations, such as constructing parallel/perpendicular lines, triangles, and various centers of triangles. Students will also watch a one-minute YouTube video explaining why the angles in a triangle sum to 180 degrees, after which they will reconstruct the diagrams used in the video in GeoGebra.

After the introduction, the instructor will explain the importance of the angle sum identities for sine and cosine (in terms of deriving other identities, which will be done in the next activity), and explain that this activity will focus specifically on the angle sum identity for *cosine*. The main part of the activity involves using a GeoGebra diagram showing points $(1, 0)$, $(\cos(\alpha), \sin(\alpha))$, $(\cos(\alpha + \beta), \sin(\alpha + \beta))$, and $(\cos(-\beta), \sin(-\beta))$. The instructor will then ask students leading questions to guide their inquiry. For example, "What can you tell me about the distances from $(1, 0)$ to $(\cos(\alpha + \beta), \sin(\alpha + \beta))$ and from $(\cos(\alpha), \sin(\alpha))$ to $(\cos(-\beta), \sin(-\beta))$? Do you notice any patterns or relationships between these distances?" This will prompt students to consider the geometry of the unit circle and the properties of these points.

As students discuss their observations, the instructor will guide them to notice that both pairs of points involve angles of $\alpha + \beta$ and that they are on the unit circle. Further questions like, "Why do you think these distances might be important? How could we verify if these distances are equal?" will encourage students to think about using the distance formula. The instructor will facilitate their exploration, prompting them to calculate these distances and set the equations equal to each other. This process will help them derive the angle sum identity for cosine: $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

Following this, students will engage in open-ended exploration. They will brainstorm ideas for how to derive the angle sum identity for *sine* by, for example, coming up with a different possible diagram, and will share their ideas with the class. The lesson concludes with a class discussion where students discuss their findings and any difficulties they encountered. The instructor will summarize the importance of the angle sum identities and their applications in trigonometry.

For assessment, the instructor will collect and review the worksheets to evaluate students' understanding and correctness of their derivations. Additionally, student participation in discussions and engagement with the GeoGebra activity will be observed to gauge their comprehension and involvement in the lesson.

GeoGebra Instructions: Getting Started

Go to the GeoGebra website: https://geogebra.org and click on “Start Calculator” or download and open the GeoGebra app. Switch from the “Graphing” app to the “Geometry” app from the list of available options at the top. To draw a line, select the “Line” tool from the toolbar. Click on the canvas (i.e., the big white part of the screen) to place the first point, then click again to place the second point, creating a line. To draw a line segment, select the “Segment” tool and click two points on the canvas. The segment will appear between them. To construct a triangle, select the “Polygon” tool, then click on three points on the canvas and click back on the first point to complete the triangle. Use the “Move” tool to move objects (points, etc) around. For example, you may drag a point to move the triangle by first clicking on the “Move” tool and then dragging the point. Click the “More” option to see more available tools, such as “Midpoint or Center,” “Perpendicular Line,” etc. To measure an angle in the triangle we have constructed, select the “Angle” tool, and click on the three points in order (the vertex should be the second point that is clicked). The angle measure will appear on the canvas. If we want to draw a line that is parallel to another line that is already on the canvas, click on the “Parallel Line” tool, then click on both the line that you want the new line to be parallel to as well as a point on the canvas that you want the line to pass through. The same can be done for a perpendicular line using the “Perpendicular Line” tool. Many other things can be done on GeoGebra, and the instructor is encourage to spend some time exploring before presenting this activity.

MIP Components of Inquiry

Active Learning

This activity will engage students in active learning by guiding them through the process of deriving the angle sum identity for cosine using GeoGebra. Rather than being presented with the identity, students will use GeoGebra to explore and discover the relationships between points on the unit circle. Through leading questions, students will be encouraged to make observations and use the distance formula to derive the identity themselves. This hands-on, inquiry-based approach ensures that students are actively involved in the learning process, enhancing their understanding and retention of the concepts. By manipulating points on the unit circle and calculating distances, students will engage in a dynamic exploration that reinforces their geometric and algebraic skills.

Meaningful Applications

This activity emphasizes meaningful applications by connecting the derivation of the angle sum identity for cosine to geometric ideas involving the unit circle. Students will use GeoGebra to visualize points and distances on the unit circle, helping them see the relevance of trigonometric identity. The visualization of geometric relationships on the unit circle and the application of the distance formula highlight the interconnectedness of algebra and geometry, providing a deeper understanding of the importance of trigonometric identities in solving complex problems.

Academic Success Skills

This activity promotes academic success skills by encouraging students to engage in critical thinking and problem-solving. By working through the derivation process, students will develop their ability to analyze geometric relationships and apply algebraic techniques. The collaborative nature of the activity, where students discuss their observations and share their ideas, also enhances their communication skills and fosters a supportive learning environment. Ultimately, this inquiry-based approach helps students build confidence in their mathematical abilities and see themselves as capable problem-solvers. Additionally, the activity promotes perseverance and attention to detail as students navigate through the process of discovery and derivation, reinforcing their overall academic skills.