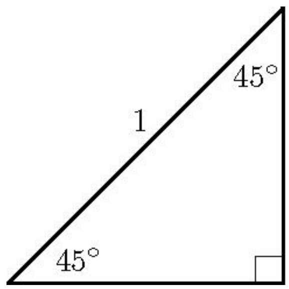
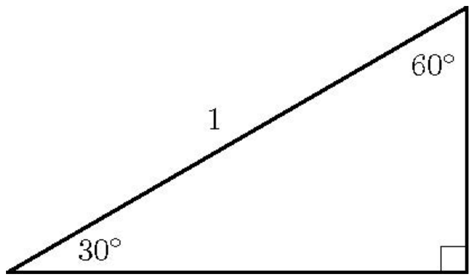
Exploring Special Triangles

In this activity, we will explore three "special" right triangles:

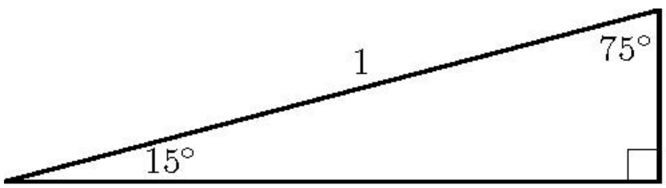
45-45-90:



30-60-90:



15-75-90:



Our goal is to find the missing side lengths. We assume the hypotenuse has length 1 in each case.

Estimation:

1. Estimate the lengths of the two legs of the **45-45-90 triangle** (see picture above). Hint: Is the bottom leg 50% as long as the hypotenuse? 60%? 75%? Make an estimate.

   Record your estimates for the lengths of the two legs (expressed as decimals):

   Answers will vary. Possible answers are:
   0.7 0.7

2. Now, estimate the lengths of the two legs of the **30-60-90 triangle** (see picture above):

   Record your estimates for the lengths of the two legs (expressed as decimals):

   Answers will vary. Possible answers are:
   0.8 0.5

3. Finally, estimate the lengths of the two legs of the **15-75-90 triangle** (see picture on previous page):

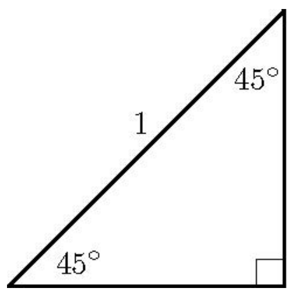
   Answers will vary. Possible answers are:

   0.9   0.25

Derivation:

4. Now, find the exact lengths of the legs of the **45-45-90 triangle**.

   Hint: Label the sides. Think about symmetry and use any relevant theorems.



$$a^2 + a^2 = 1^2$$

$$2a^2 = 1$$

$$a^2 = \frac{1}{2}$$

$$a = \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Record your answers here (if there are square roots, rationalize the denominator so that we all have a uniform way of writing the answer):

$\frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{2}$

Include decimal approximations (to 3 decimal places) for the two leg lengths found above:

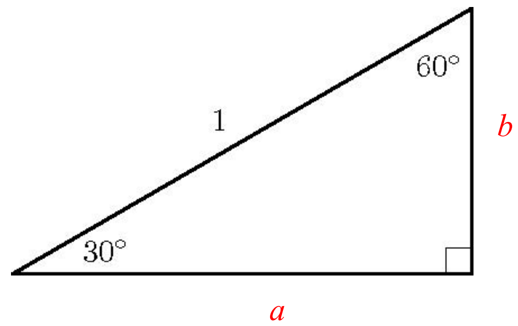
0.707   0.707

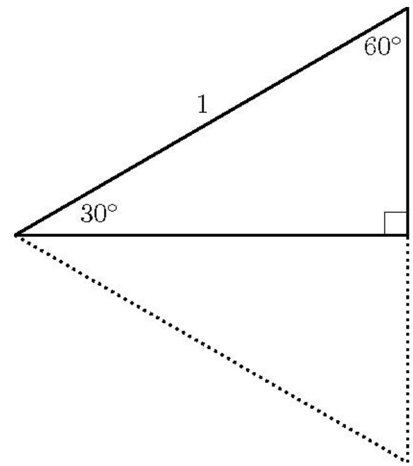
Compare your earlier estimates with the derived lengths. Discuss any discrepancies with your group.

Answers will vary. Estimates that were between 0.6 and 0.8 are reasonable.

5. Find the <u>exact</u> lengths of the legs of the **30-60-90 triangle**.

<u>Hint</u>: Label the sides. Think about symmetry/reflection and use any relevant theorems. Can you make a larger triangle from two copies of this one? If you get stuck, ask your instructor for a hint.





By reflecting across the bottom leg, we obtain a large equilateral triangle (see diagram on the right). Since the sides of an equilateral triangle all have the same length (in this case length 1), we have that *b* is equal to ½. We then use the Pythagorean Theorem to obtain:

$$a^2 + b^2 = 1^2$$

$$a^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$a^2 + \frac{1}{4} = 1$$

$$a^2 = \frac{3}{4}$$

$$a = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$$

Record your answers here (if there are square roots, rationalize the denominator so that we all have a uniform way of writing the answer):

$\frac{\sqrt{3}}{2}$ $\qquad\qquad$ $\frac{1}{2}$

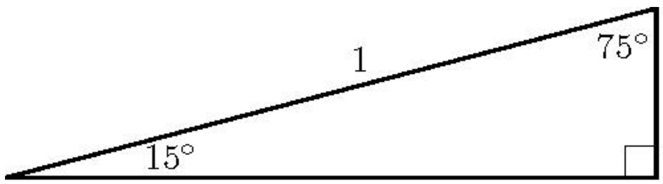
Include decimal approximations (to 3 decimal places) for the two leg lengths found above:

0.866 $\qquad\qquad$ 0.5

Compare your earlier estimates with the derived lengths. Discuss any discrepancies with your group.

Answers will vary. For the longer leg, estimates between 0.75 and 0.95 are reasonable. For the shorter leg, estimates between 0.4 and 0.6 are reasonable.

6. Find the exact lengths of the legs of the **15-75-90 triangle**.



This problem is a bit more challenging! Try your best with your group to figure it out!

See the "Hint #2 diagram" on page 6. In this diagram, the sides of the 30-60-90 triangle are $\frac{\sqrt{3}}{2}$ and $\frac{1}{2}$ since the hypotenuse is 1. The largest (outside) triangle is a 45-45-90 triangle. The top (dotted) triangle is then also a 45-45-90 triangle, so its legs have the same length. One of those legs is $\frac{1}{2}$, so the other leg is also $\frac{1}{2}$. The hypotenuse has length $\sqrt{2}$ times the length of these legs, so that the hypotenuse has length $\frac{\sqrt{2}}{2}$. We also see now from the diagram that the hypotenuse of the outside 45-45-90 triangle has length $\frac{\sqrt{3}}{2}+\frac{1}{2}$. The legs of this triangle have lengths that are $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse (since it is a 45-45-90 triangle), meaning that the legs have lengths $\frac{\sqrt{2}}{2}\cdot\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)=\frac{\sqrt{6}+\sqrt{2}}{4}$. This means that the lower (horizontal) leg of the 15-75-90 triangle (i.e., the one opposite the 75-degree angle) has length $\frac{\sqrt{6}+\sqrt{2}}{4}$. We may then use the Pythagorean Theorem $a^2+\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2=1$ to obtain the length of the other leg. Solving for a, we obtain $a=\frac{\sqrt{6}-\sqrt{2}}{4}$.

Record your answers here (if there are square roots, rationalize the denominator so that we all have a uniform way of writing the answer):

$\frac{\sqrt{6}+\sqrt{2}}{4}$ $\frac{\sqrt{6}-\sqrt{2}}{4}$

Include decimal approximations (to 3 decimal places) for the two leg lengths found above:

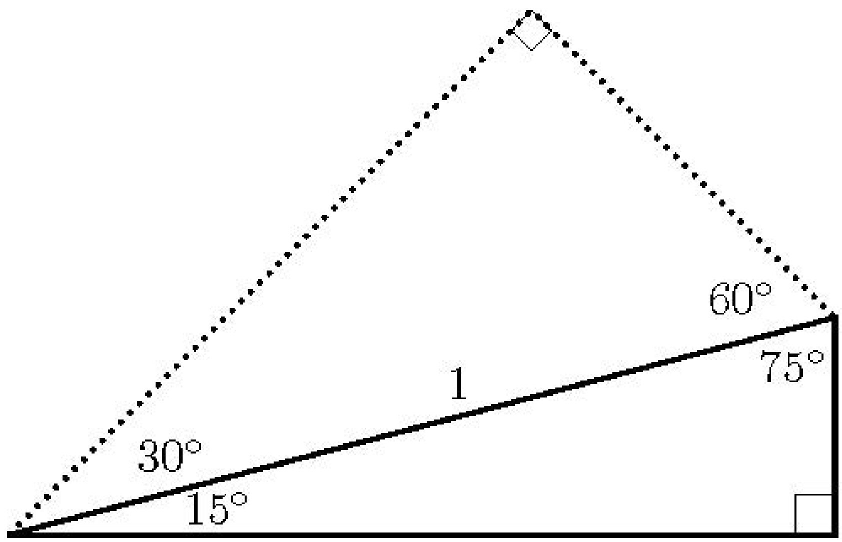
0.966 0.259

Compare your earlier estimates with the derived lengths. Discuss any discrepancies with your group.

Answers will vary. For the longer leg, estimates between 0.9 and 0.99 are reasonable. For the shorter leg, estimates between 0.15 and 0.35 are reasonable.

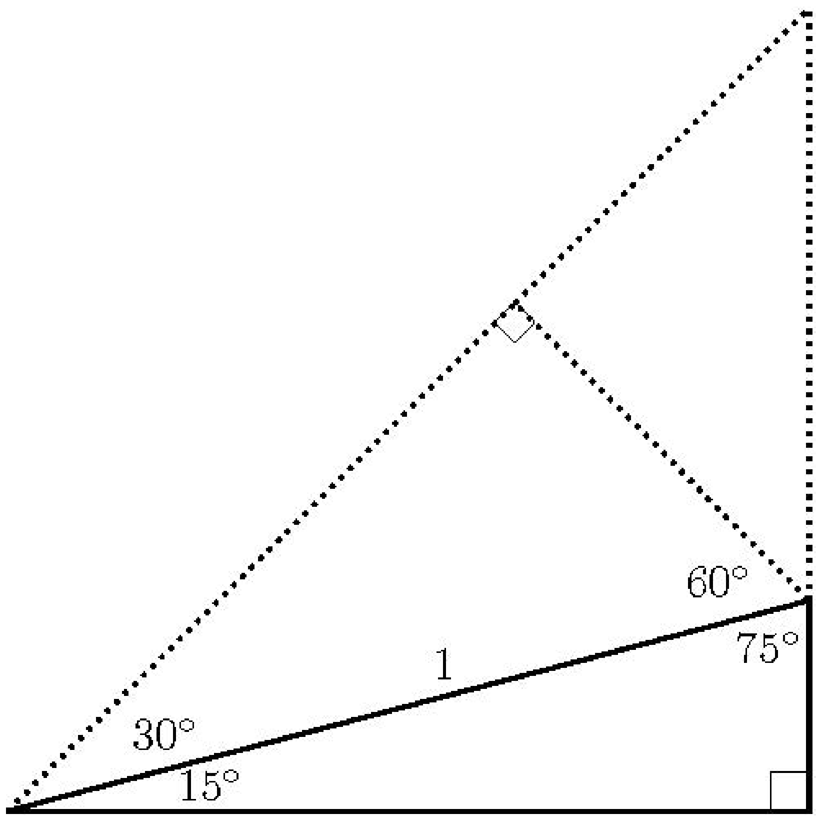
Note: This diagram should only be given to students after they have wrestled with the problem for a few minutes.

Hint #1



Note: This diagram should only be given to students if they are unable to make progress using "Hint 1" after several minutes.

Hint #2



Application Challenges:

(a) A ladder leans against a wall forming a 45° angle with the ground. If the ladder is 10 feet long, how high does the ladder reach up the wall? Round to 2 decimal places.

$$10 \cdot \frac{\sqrt{2}}{2} \approx 7.07 \text{ ft.}$$

Solution: ≈ 7.07 ft.

(b) A flagpole casts a shadow forming a 30° angle with the ground. If the length of the shadow is 52 feet, how tall is the flagpole? Round to 2 decimal places.

$$53 \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{2} = \frac{53}{\sqrt{3}} \approx 30.02 \text{ ft.}$$

Solution: ≈ 30.02 ft.

(c) An architect is designing a triangular window that forms a 30° angle at the peak. The base of the window is 6 feet long. What is the height of the window? Round to 2 decimal places.

$$3\left(\frac{4}{\sqrt{6}-\sqrt{2}}\right)\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) \approx 11.20 \text{ ft.}$$

Solution: ≈ 11.20 ft.

Discuss the solutions to the application problems with your group and compare your approaches. Reflect on how the properties of the special triangles facilitated your problem-solving process.