LESSON TITLE: Exploring Geometric Series

OVERVIEW: This activity introduces students to geometric series through geometry, algebra, hands-on exercises, and real-world applications. It aims to provide students with an introduction to the idea of convergence of an infinite series, which is particularly useful for students progressing to Calculus. The activity also includes a bonus homework extension which explores the relationship between centroids of triangles and a geometric series.

PREREQUESITE IDEAS AND SKILLS:

- Students should have basic algebra skills, including solving equations, simplifying expressions, adding rational expressions, and dividing by polynomials.
- Students should understand exponents and (optionally) series notation.
- Students should be familiar with basic geometric concepts such as area and length.
- For the bonus homework extension, students should understand that the three medians of a triangle (i.e., the lines that connect a vertex to the midpoint of the opposite side) meet at a point called the "centroid" of the triangle, and that these three medians divide the triangle into six smaller triangles that have equal area.

MATERIALS NEEDED TO CARRY OUT THE LESSON

- Activity Worksheet
- Pieces of string (or dental floss or fishing line) for each group (one meter long)
- Scissors
- Calculator
- Graph paper and pen/pencil for geometric interpretations

CONCEPTS TO BE LEARNED/APPLIED

- Students will understand that geometric series can be represented and visualized with diagrams, fractions, and repeating decimals.
- Students will derive the formula for the sum of a geometric series and will use it to sum various geometric series and to write a repeating decimal as an infinite series.
- Students will analyze and solve real-world application problems involving geometric series, such as the total distance traveled by a bouncing ball.
- Students will connect algebraic expressions to geometric representations, deepening their comprehension of series convergence.

INSTRUCTIONAL PLAN

Start the lesson with an introduction to the main question: "Is it possible to add infinitely many positive numbers and NOT get infinity as the sum?" This will engage students and set the stage for exploration of geometric series.

Hand the activity worksheet to the students and ask them to observe the series formulas at the beginning of the worksheet and to answer the question below them. It might be helpful to ask them to plug the first few terms of one of the series into a calculator to find the sum and to see that it gets "closer" and "closer" to a certain number. For example, $\frac{1}{3}+\frac{1}{9}=0.4444$. Then $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}=0.481481$ ….. Then $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}=0.493827$ …. In this way, they can observe that it begins to approach $0.5=\frac{1}{2}$ as more terms are added.

Part 1: Hands on Exploration with String

Divide students into groups of three and provide each group with a one-meter piece of string. One student (Person A) in each group cuts the string into thirds and gives one piece to each of the other two students (Persons B and C). Person A then continues to divide their remaining third into thirds again, giving one piece to each of the other students and keeping one for themselves. This process continues until the string is too small to be cut again. (Note: dental floss or fishing line might work better than string to cut into very small pieces.) Students should observe and record what happens after many iterations, noting the Persons B and C each get closer to having half the string. This illustrates the infinite series:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

Next, have students form groups of four and repeat the process with Person A dividing the string into fourths. Students should observe that Persons B, C, and D each get closer and closer to having one-third of the string, illustrating the series:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\frac{1}{3}$$

Explain that these are examples of geometric series, where each term after the first is found by multiplying the preceding term by a fixed non-zero number called the common ratio.

Part 2: Algebraic Exploration

Guide students through Part 2 of the worksheet. They may need help with the first question of how to simplify the expression $(1-x)(1+x+x^2+x^3+x^4)$. Tell them that each term within the first set of parentheses needs to be multiplied by each term in the second set. They can do this in a systematic way by:

- multiplying 1 by each term in the second set of parentheses
- then multiplying $-x$ by each term in the second set of parentheses

Students should observe the "telescoping" nature of the expansion. Students are then asked to generalize these results. In the next several questions, students may need some help understanding what it means to let n approach infinity, and why x^{n+1} approaches 0 when n approaches infinity, if we assume that $|x|<1$. You can give examples such as, "What happens if we take a number and then repeatedly multiply by a positive number less than 1 (such as $\frac{1}{2}$ or $\frac{2}{3}$)?" They should notice that it will get closer and closer to 0. This is an early introduction to the idea of a "limit," so the discussion should be informal and does not need to be rigorous. You should also mention what happens when x is greater than 1, such as when $x=5$. This again provides an opportunity to give a brief (informal) discussion of the concept of "divergence."

Ultimately students derive the following formulas (valid for $|x| < 1$).

$$1 + x + x^2 + x^3 \ldots = \frac{1}{1 - x}$$

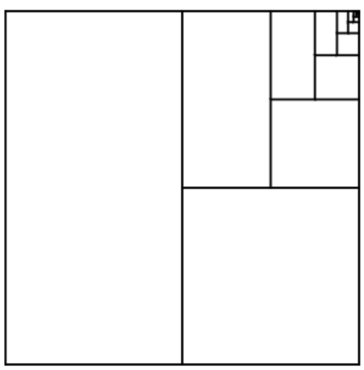
$$x + x^2 + x^3 + \cdots = \frac{x}{1 - x}$$

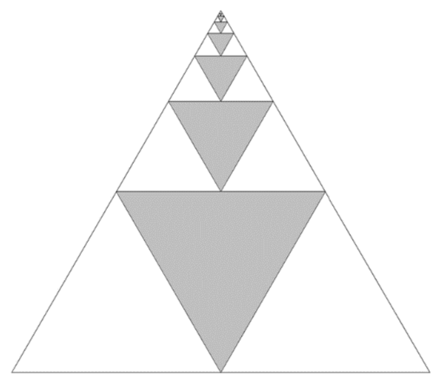
Depending on the level of the students, the instructor can provide hints/help with the algebra involved in these derivations.

Students can then plug in their own values (such as $\frac{1}{2}, \frac{3}{4}$, or $-\frac{1}{10}$) into these formulas to obtain the sums of various geometric series.

Part 3: Geometric Exploration

Introduce geometric representations of geometric series. The following diagrams are on their worksheet, and the instructor can write them on the board as well. It may be helpful to have the instructor walk students through showing that the first diagram.





To do this, tell students to think of the square as having an "area" of 1. Shade in half of the square (the left-half in the diagram). Then shade in half of what remains (so $\frac{1}{4}$ of the original square). Then half of what remains (so $\frac{1}{8}$ of the original square). In this way, students can see that the total area shaded is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

But after infinitely many iterations, the <u>whole</u> square is filled (which has an area of 1). We have therefore illustrated the geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Students should then try to discover which geometric series the second diagram (the triangular one) illustrates. The answer is:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\frac{1}{3}$$

The following excellent YouTube video has many other examples: Click here.
The instructor may choose to either:

1. Show the video to the class.

2. Pick example diagrams from the video and ask students to discover which geometric series the diagram illustrates. This is actually quite fun!

If time permits, students should also be encouraged to come up with their own example diagrams that illustrate various series. Creativity is encouraged!

Part 4: Repeating Decimals

The activity worksheet then gives an example of the repeating decimal 0.11111… and asks students to write it as an infinite geometric series and to sum the series to get 1/9. They should obtain:

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\cdots=\frac{1}{9}$$

Part 5: Real-World Application (Bouncing Ball)

As a practical application, students can analyze a bouncing ball scenario:

> A ball is dropped from a height of 6 feet and bounces back to three-fourths of its previous height. Students will calculate the total distance traveled by the ball after infinitely many bounces.

By considering the ball bouncing down, then up, then down, then up, etc., the instructor can give hints to guide students to formulate the series:

$$6+6\left(\frac{3}{4}\right)+6\left(\frac{3}{4}\right)+6\left(\frac{3}{4}\right)^2+6\left(\frac{3}{4}\right)^2+6\left(\frac{3}{4}\right)^3+6\left(\frac{3}{4}\right)^3+\cdots$$

which equals

$$6+12\left(\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\cdots\right)=6+12\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right)=6+12\times 3=42$$

The ball thus travels a total of 42 feet.

The activity worksheet asks students to see if they can come up with their own real-world examples of infinite series.

Throughout the lesson, the instructor should facilitate group discussions, provide individual support as needed, and encourage students to actively engage with the material. Completed worksheets, group presentations, and class discussions should be used to assess understanding and provide feedback.

Active Learning

This activity engages students in active learning through discovery and exploration of geometric series. Students begin by wrestling with the question of whether it is possible for infinitely many positive numbers to be added together to get a sum which is not infinite. Students will select addition of different amounts of finite terms, perform the arithmetic, and evaluate what the results approach. They perform a hands-on exploration involving dividing a piece of string, which provides a tangible (and fun) connection to the abstract concept of infinite series, and they perform an analysis of what the string division activity means mathematically, in terms of series. When cutting string, students will select various numbers of finite terms that represent the mount of string Person B and Person C have, perform the arithmetic, and evaluate what the result approaches. Students also use algebra to derive the formulas for both a finite and infinite geometric series, and they select various values to be applied to the formula for an infinite geometric series. The geometric interpretations involving squares and equilateral triangles offer a visual and interactive way to understand these series and the concept of convergence. For the geometry, students will select an area for the starting triangle (likely 1), perform an analysis to determine the sizes of the pieces, and evaluate the patterns in the sizes they see. For the part of the activity that involves repeating decimals, students will select a representation (i.e., fractions or decimals), perform computations, and evaluate whether the representation lets them see the pattern. If not, they will select a different representation and see if that choice lets them see the pattern. This will help students learn that sometimes different representations afford different insights into mathematical phenomena. These tasks promote active engagement as students make predictions, test their hypotheses, and derive results through guided inquiry, rather than passively being given formulas.

Meaningful Applications

The activity emphasizes meaningful applications of geometric series by connecting abstract mathematical concepts to tangible and visual examples. The string division part of the activity illustrates how geometric series can model a real-world physical process. The geometric division of squares and triangles (and other figures) gives a mathematically meaningful visual representation of series convergence. With these visuals, students discover how it is possible that infinitely many positive numbers can be added to get a sum which is not infinite: they “see” how the convergence is possible. Additionally, the real-world application involving a bouncing ball shows the practical use of geometric series in calculating total distances, thus linking a theoretical understanding to a relatable context. This approach helps students see the relevance of geometric series in both mathematical theory and everyday situations.

Academic Success Skills

This activity promotes academic success skills by encouraging students to discover and derive mathematical concepts independently. By actively engaging in algebraic manipulations, geometric interpretations, and hands-on exploration, students develop critical thinking and problem-solving skills. They learn to make and test conjectures, analyze patterns, and generalize their findings. The collaborative nature of the exercises, such as the string division task and group problem-solving, enhances their ability to communicate mathematical ideas effectively. This inquiry-based approach empowers students to take ownership of their learning, build confidence in their mathematical abilities, and apply their knowledge in diverse contexts.