**Building Understanding of the Epsilon-Delta Definition of a Limit via Graphic Representations**

**Relevant Course:** Calculus I

**Estimated Time for Lesson:** 100 minutes *–* The investigation is designed to take about two class periods and can be done individually, in small groups, or as a class. We recommend that part of the activity be completed in class and part of it outside of class, in order to ensure time to explore all necessary content in a Calculus I course and to put more responsibility for learning on the students and afford them ample time to complete the tasks and reflect on their investigations.

**Compatible Instructional Formats:** Whole-class, small group, or individual investigations (Access to one of *Desmos*, *The Geometer’s Sketchpad*, or *GeoGebra* is needed, which may require Internet access.)

**Activity Overview:** The proposed inquiry lesson is intended to support students in constructing meaningfulness for the ε-δ definition of the limit of a function and to accomplish that through the manipulation, analysis and interpretation of graphic representations created in *Desmos* or the dynamic geometry software packages *GeoGebra* or *The Geometer’s Sketchpad*, with the latter now being available as a free download (<https://dynamicmathematicslearning.com/free-download-sketchpad.html>).  Particular emphasis is placed on connecting components of the graph to the ε-δ definition of the limit. Within the lesson the students or an instructor will manipulate a dynamic electronic sketch depicting a function. Values for a proposed limit *L* of a function and a value *a* that approaches, and a point on the function of interest can each be altered by either dragging a point in the graph or a slider. Students or instructors will then be able to apply the definition of a limit to determine whether or not for a chosen ε, does a given δ value determine a neighborhood about *a*, such that if an value is within δ units of *a*, must the corresponding *f(x)* or *y*-values be within ε units of *L*? With that question in mind and with the aid of dynamic sketches, students will explore whether or not it is possible to find *f(x)* or *y-*values that lie outside an open rectangular area, if the -values that yield the pertinent *f(x)* values lie within δ units of *a*. If not, students or instructors can adjust the value of δ to see if such adjustments yield an open neighborhood about *a* such that the *f(x)* values for the *x*-values in that neighborhood of *a* lie within the open rectangular region, i.e., are within ε units of the proposed limit *L*. Through investigations such as these, the lesson will nurture students’ construction of deep, conceptual understanding of the ε-δ definition of the limit of a function and allow them to interpret that definition from a graphic or geometric perspective as well as an algebraic perspective. This will also allow pupils to think about the limit of a function both more flexibly and as more than a number a function gets close to as *x* approaches some given value. Accordingly, the lesson can be used in a Calculus I course, and a variation on it could be used in later calculus coursework when exploring potential convergence of sequences or series.

The stage for the activity can be set via the following problem, which can initially be explored and used as a steppingstone to constructing meaningfulness for the ε-δ definition of a limit.

As in the *Martian*, a crew of astronauts has plotted a course for their spacecraft, which will bring them in close proximity to a second spacecraft and allow the crew of the second vehicle to come across on 10-meter tethers. If the distances above minimal rescue height for the two spacecraft, with respect to the zenith point directly above the second crew’s launch point are:

*Rescue Vehicle:*

*Second Vehicle:*

What set of -values will allow the spacecrafts to come within 10 meters of each other?

Note, = kilometers from zenith for launch point, and and yield kilometers above minimal rescue height for the spacecrafts’ respective trajectories.

After using a dynamic electronic sketch to explore the above introductory question, the instructor can note the similarity of the graphic solution process to using the ε-δ definition of the limit of a function to determine whether or not the limit of a function exists as approaches a given value *a*. The instructor can then engage the whole class in a Socratic dialogue, or the instructor can engage small groups of students or individual students in determining whether or not the limit exists, and if it exists determine its value and use the graph and the ε-δ definition of limit to justify their assertion. We recommend that instructors have students do this for each of the following functions:

Note that *Geometer’s Sketchpad* does not represent the last two examples well, so those two examples are not included in the set of sketches constructed with *Sketchpad*.

In each case, the students and/or instructor choose values for *L, ε, a,* and *δ*. Corresponding points or sliders in a dynamic sketch are then dragged to reach the selected values. The investigators then consider the question: If an *x* coordinate is within *δ* units of point *a* on the *x-*axis, is or must its related *f(x)* or *y* coordinate lie within *ε* units of *L*? This is determined by considering whether or not the *f(x)* or *y* coordinate of a moveable point *P* must fall in the open rectangle determined by the points

*,* and, if *P*’s *x* coordinate is within δ units of point *a*. If the answer is yes, pupils will be able to drag point *P* along the graph of *f(x)* and see that any time *P* lies between the vertical lines and *, P* will also lie inside the open rectangle noted above, or to put it another way, students will see that *P* also lies between the horizontal lines and

. If the answer is no, students will be able to drag *P* to a location on function *f(x)* such that *P* lies between the vertical lines and *,* but outside the open rectangle or outside the horizontal lines and . Additionally, if the answer is no, pupils consider whether or not the answer can be changed to yes, by changing the value of δ (by moving a slider) to obtain an open rectangle which does contain the *f(x)* value for any *x*-value lying within δ units of point *a*. Again, this is determined by dragging point *P* through the region and observing its location with respect to the horizontal and vertical lines that contain the segments that bound the relevant open rectangle.

Dynamic sketches for the introductory problem and each of the additional examples have been provided in three formats, *Desmos, The Geometer’s Sketchpad, and GeoGebra*, with the exception of 2 functions that were not represented well in *Sketchpad*.. Accordingly, we have provided a total of 25 sketches across three formats to make it as easy as we can for instructors to use the tool that best fits the resources and constraints of their personal circumstances. Some notes on how to manipulate key points or values in the dynamic sketches follow:

In sketches created with *The Geometer’s Sketchpad*, you can alter epsilon (E) or delta (d) by dragging the appropriate slider. You can alter *a* or *L* by dragging the related points on the *x*- and *y*-axis, respectively. For zooming in or out, click and drag the neon green point to the right to zoom in or to the left to zoom out.

In *Desmos* sketches, to alter epsilon (E) or delta (d), drag the appropriate slider. You can change *L* by either dragging its slider or by dragging point L on the *y*-axis. To alter *a*, drag the slider for point *a* rather than point *a* in the sketch. You can also alter *a*’s position by dragging the point labeled (*a*, f(*a*)) in the sketch, although due to constraints of the software the labels will appear as “a” and “(a, f(a)).”

In *GeoGebra* sketches, to alter epsilon (ε) or delta (δ), drag the appropriate slider. You can change *L* by either dragging its slider or by dragging point L on the *y*-axis. Similarly, you can alter *a* by dragging its slider or by dragging point on the *x*-axis, although due to constraints of the software the label will appear as “a” rather than .

We have included several examples because we believe it is important for students to see a variety of cases, and that the initial example or two should entail functions that are simple and familiar enough to students that they do not have to invest substantive mental energy to make sense of the given function and likewise, do not have to consider any complicating factors such as holes or other discontinuities. This will allow students to invest all of their mental energy in making sense of the ε-δ definition of the limit. With the aid of the dynamic representation of the first example or two, students can come to understand that for a chosen ε, there may be some δ-values for which it is possible to enclose all of the *f(x)* values for *x*-values “near” *a* within an open, rectangular neighborhood whose height is an open interval centered about a proposed limiting value *L* on the *y*-axis, running from *L* – *ε* to *L* + *ε*, and whose base is an open interval centered about *a* on the *x*-axis, running from *a – δ* to *a* + *δ* . Similarly, students should realize that for some values of δ, even if *x* is within δ units of *a*, it is not possible to embed all corresponding *f(x)* values in the noted open rectangle, i.e., to ensure that related *y*-values are within ε units of *L*. Of course, attention must be given to highlighting the fact that stating *L* is the limit of *f(x)* as *x* approaches *a*, means that for any selected value of , it is possible to find a such that if is within units of *a*, then must be within ε units of *L*. As previously indicated, we believe it is important to cover the range of examples well when teaching mathematical concepts and processes. Accordingly, we have built dynamic sketches for examples for continuous functions, for functions with removable discontinuities, for piecewise functions with jump discontinuities, and for functions that are not piecewise defined but do have jump discontinuities. We recognize that opinions may vary with respect to the range of examples to which students should be exposed to construct the necessary depth of conceptual understanding and procedural fluency to progress in their study of calculus. Herein, we have simply opted to provide instructors with the option of starting with very simple examples and gradually progressing to more complex examples that aggregately span the range of possible cases reasonably well.

**Primary Conceptual Goals:** This lesson for teaching the ε-δ definition of a limit from a graphic perspective is intended to support students in attaining the following conceptual goals:

* Students will build understanding of the concept of a limit from geometric perspective, and thereby, be able to understand that for an open interval of any radius ε chosen about a given point *L* on the *y* or *f(x)* axis, a radius δ can be found about the *x* coordinate of that *f(x)* or *y-*value, so that if the *x* coordinate is within the open interval about *x* on the *x* axis, the related *f(x)* or *y*-value is also inside the open interval around *L* on the *y* or *f(x)* axis.
* Students will understand that geometric or graphic representations can augment understanding of algebraic representations and highlight relationships, meanings and constraints that may not be as easily or as well-revealed or understood via the use of an algebraic representation alone.
* Students will construct a deep and flexible understanding of the concept of limit that will serve as a foundation on which they can build similarly deep and flexible understandings and use both graphic and algebraic representations to give meaning to the concepts of derivative and integral.

**Interrelating the *ε*-*δ* Definition of a limit & Other Calculus Topics:** We believe that activities such as those proposed herein will help students construct a geometric / graphic understanding (a mental image or representation) of the concept of a limit that highlights the roles that selected values of ε and δ play within that image and related definition. Moreover, since by definition, both derivatives and integrals are limits, a geometric or graphic conception of the limit that provides a visual, mental structure (a cognitive framework) that can also be used as a foundation on which students can build understanding of the concepts of derivatives and integrals, again with geometric representations serving as tools to give meaning to those concepts. In the case of the derivative, the concept could even be envisioned as the limit of a sequence of slopes of secant lines, with that limit being the slope of the tangent line at the point of interest. Likewise, in the case of the integral, the concept could be envisioned as the limit of a sequence of Riemann sums of approximations of the area between the graph of a function and the *x*-axis, with that limit being the value of the integral for the region in question. Such methods of conceiving of a derivative or an integral would help students understand that, if the derivative or integral exists, then for any chosen value of ε, it is possible to find a sufficiently small δ to satisfy the relevant limit definition. Of course, each of these geometric approaches can also be viewed as or connected to a numerical sequence that has a limiting value, like say the sequence of ratios of consecutive Fibonacci numbers, given by *Fn / Fn-1*, which converges to the Golden Ratio. Moreover, the fact that the derivative can be viewed as the limit of a sequence of slopes of secant lines or the limit of a sequence of average rates of change and the integral as the limit of a sequence of area approximations provide students with a foundation on which they can build subsequent understanding of limiting values of sequences and the convergences or divergence of series in later calculus or analysis courses.

**Active Learning, Meaningful Applications & Academic Success Skills:** The proposed lesson reflects the principle of *active learning* in that it necessitates that students actively engage their minds in an effort to give meaning to graphic representations and a symbolic, algebraically oriented definition that necessitates construction of interrelationships between those representations. Construction of those interrelationships in turn constitutes the foundation on which deep, rich, interwoven understandings of the concept of a limit of a function are then built. In the process of doing all the above, the students select, perform, and evaluate the actions and related mathematical structure which equate to the structures of the concepts to be learned. All of this similarly requires that students *meaningfully apply* their knowledge of mathematical structures such as variable, graph, function, neighborhood, graphic representations, and logical reasoning skills that permeate mathematics and mathematical problem solving. Moreover, learners must give meaning to both the previously noted mathematical structures and the algebraic symbolism inherent to the definition of the concept of limit by relating them to graphic or geometric representations. These meanings and interpretations are then used as the basis to make, to some extent justify, and to generalize the results and requirements of the definition of a limit across contexts based on identified or extracted mathematical structure. Finally, the proposed activity reflects the principle of facilitating *academic success skills* by engaging students in activities through which they can do what mathematicians do, and in the process, experience the joy of mathematical creation while constructing deep understanding on which procedural fluency can be built. Moreover, that conceptual understanding and procedural expertise will allow students to apply their learning, all of which enhances students’ self-esteem as mathematical practitioners, as well as their appreciation of and motivation to learn about and participate in the field.

**Student Directions: Understanding the Epsilon-Delta Definition of a Limit via Graphs**

**Introduction Problem:** Save the Crew

As in the *Martian*, a crew of astronauts has plotted a course for their spacecraft which will bring them in close proximity to a second spacecraft to allow the crew of the second vehicle to come across on 10-meter tethers. If the distances above minimal rescue height for the two spacecraft, with respect to the zenith point directly above the second crew’s launch point are:

*Rescue Vehicle:*

*Second Vehicle:*

What set of -values will allow the spacecrafts to come within 10 meters of each other?

Note, = kilometers from zenith for launch point, and and yield kilometers above minimal rescue height for the spacecrafts’ respective trajectories. Explain how the graph supports your answer.

The preceding task is similar in nature to applying the definition of a limit to determine if a given function *f(x)* has a limit as *x* approaches a value *a,* and the concept of a limit pertains to several other key concepts in calculus, such as the derivative, integral, and sequence convergence. For that reason, it is very important that you have a deep understanding of limits. Accordingly, in this activity, you will manipulate, analyze and interpret graphs to construct a visual representation of what the symbolic definition of a limit means. Specifically, you will manipulate components of electronic sketches and apply the definition of a limit of a function to determine whether or not the limit of a given function exists as *x* approaches some value *a*, and if so, state its value and explain why the limit exists. That is, you will explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

Definition of a Limit: The is equivalent to stating that for any , there exists a , or it is possible to find a , such that if then , i.e., such that if is within units of but not equal to , must be within units of .

For each of the given functions, you will be given or you will choose a value for *a*, propose a value for the limit *L* as *x* approaches *a*, choose or be given one or more values for ε, for each value of ε determine if it is possible to find a value for δ, such that if an *x* coordinate is within δ units of *a*, mustthe corresponding *f(x)* values lie within ε units of L. Then you will explain why *L* is or is not the limit of *f(x)* as *x* approaches *a*. That is, you will explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit. Keep in mind that you may have to try multiple values for δ.

1. Take .
2. Find if possible, and state it here.
3. Position the point labeled on the -axis at the location from part a.
4. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
5. Position the point labeled on the -axis at the location chosen in the preceding step.
6. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
7. Does the exist? If so, what is it?
8. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
9. For each value of from part f, what is the largest value of that will work?
10. Find a general expression for the largest value of for a fixed but arbitrary , i.e., express in terms of .
11. Take .
12. Find if possible, and state it here.
13. Position the point labeled on the -axis at the location from part a.
14. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
15. Position the point labeled on the -axis at the location chosen in the preceding step.
16. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
17. Do you think the exist? If so, what is it?
18. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
19. For each value of from part f, what is the largest value of that will work?
20. Take .
21. Find if possible, and state it here.
22. Position the point labeled on the -axis at the location from part a.
23. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
24. Position the point labeled on the -axis at the location chosen in the preceding step.
25. Choose three values for between and . If possible, for each value for ε, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lies within units ofon the -axis.
26. Do you think the exist? If so, what is it?
27. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
28. Take , and repeat for *a* = 2.
29. Find if possible, and state it here.
30. Position the point labeled on the -axis at the location from part a.
31. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
32. Position the point labeled on the -axis at the location chosen in the preceding step.
33. Choose three values for between and . If possible, for each value for ε, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
34. Do you think the exist? If so, what is it?
35. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
36. For , suppose that we only wanted to consider values for *x* that were less than or we only wanted to consider what happens to as approaches from the left, if the values were within units of but not equal to , would the corresponding values be forced to lie within of ?
37. For , suppose that we only wanted to consider values for *x* that were greater than or we only wanted to consider what happens to as approaches from the right, if the values were within units of but not equal to , would the corresponding values be forced to lie within of ?
38. Take , and repeat for *a* = 1
39. Find if possible, and state it here.
40. Position the point labeled on the -axis at the location from part a.
41. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
42. Position the point labeled on the -axis at the location chosen in the preceding step.
43. Choose three values for between and . If possible, for each value for ε, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
44. Do you think the exist? If so, what is it?
45. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
46. Take .
47. Find if possible, and state it here.
48. Position the point labeled on the -axis at the location from part a.
49. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
50. Position the point labeled on the -axis at the location chosen in the preceding step.
51. Choose three values for between and . If possible, for each value for ε, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
52. Do you think the exist? If so, what is it?
53. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
54. Take .
55. Find if possible, and state it here.
56. Position the point labeled on the -axis at the location from part a.
57. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
58. Position the point labeled on the -axis at the location chosen in the preceding step.
59. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
60. Do you think the exist? If so, what is it?
61. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.
62. Take .
63. Find if possible, and state it here.
64. Position the point labeled on the -axis at the location from part a.
65. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?
66. Position the point labeled on the -axis at the location chosen in the preceding step.
67. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.
68. Do you think the exist? If so, what is it?
69. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

**Answers to selected problems:**

1. Take .
2. Find if possible, and state it here. **Answer:**
3. Position the point labeled on the -axis at the location from part a.
4. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?

**Answer:** Yes,

1. Position the point labeled on the -axis at the location chosen in the preceding step.
2. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.

**Answer:** Respectively, δ < (Answers may vary)

1. Does the exist? If so, what is it? **Answer:** Yes,
2. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

**Answer:** A graphic based explanation follows: The limit exists because for any ε > 0, it is possible to find a value so that if then the corresponding *y* or *f(x)* value must lie in the open rectangle defined by the vertical lines and *,* and the horizontal lines and . Alternatively, we see that for each given value of in part f, using the graph, we are able to find a value for so that if the coordinate is within units of point , then the corresponding or coordinate lies within units ofon the -axis. In fact, it is possible to find a value for any value of , so that whenever then . Therefore, the limit of exists as approaches

. (Answers may vary)

1. For each value of from part f, what is the range of values that will work?

**Answer:**

1. Find a general expression for the largest value of for a fixed but arbitrary , i.e., express in terms of .

**Answer:** δ <

1. Take .
2. Find if possible, and state it here. **Answer:** *f(3*) does not exist
3. Position the point labeled on the -axis at the location from part a.
4. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?

**Answer:** Yes, about (Exact value = )

1. Position the point labeled on the -axis at the location chosen in the preceding step.
2. Choose three values for between and . If possible, for each value for ε, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lies within units ofon the -axis.

**Answer:** Take . Respectively, some values for that work are . (Answers for δ vary)

1. Do you think the exist? If so, what is it? **Answer:** Yes, about (Exact value = )
2. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

**Answer:** The limit exists because for any ε > 0, it is possible to find a value so that if

 then the corresponding *y* or *f(x)* value must lie in the open rectangle defined by the vertical lines and ,and the horizontal lines and . A more algebraic argument follows: We see that for each given value of in part f, using the graph, we are able to find a value for so that if the coordinate is within units of point , then the corresponding or coordinate lies within units ofon the -axis. In fact, it is possible to find a value for any value of , so that whenever then . Therefore, the limit of exists as approaches . (Answers may vary)

1. Take , and repeat for *a* = 2.
2. Find if possible, and state it here. **Answer:**
3. Position the point labeled on the -axis at the location from part a.
4. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?

**Answer:** For : No, not applicable; For : Yes,

1. Position the point labeled on the -axis at the location chosen in the preceding step.
2. Choose three values for between and , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.

**Answer:** For :Take . It is not possible to find respective values of .

 For :Take . Respectively, some values for that work are (respectively, ranges for δ are δ < 0.6, δ < 0.1, and δ < 0.05).

1. Do you think the exist? If so, what is it?

**Answer:** For : No, not applicable; For : Yes,

1. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

**Answer:**

For : The limit does not exist. We know this because 0 and 3 would be the only reasonable choices for *L*, and for each there are ε > 0, such that for any δ > 0, there is at least one *x* value satisfying that has an *f(x)* value that does not lie in the open rectangle defined by the vertical lines and *,* and the horizontal lines and . We can also justify our assertion that the limit does not exist by noting that for both *L* = 0 and

*L* = 3, there is at least one ε > 0 for which it is not possible to find a δ value so that if

 then . (Answers may vary)

For The limit exists because for any value of , it is possible to find a so that if then the related *y* or *f(x)* value lies in the open rectangle defined by the vertical lines and *,* and the horizontal lines and . To put it another way, we see that for each given value of in part f, using the graph, we are able to find a value for so that if the coordinate is within units of point , then the corresponding or coordinate lies within units ofon the -axis. In fact, it is possible to find a

 value for any value of so that whenever then . Therefore, the limit of exists as approaches . (Answers may vary)

1. For , suppose that we only wanted to consider values for that were less than or we only wanted to consider what happens to as approaches from the left, if the values were within units of but not equal to , would the corresponding values be forced to lie within units of ?

**Answer:** Taking *L* = 0, for any , it is possible to find a , such that if *x* < 0 and

 the corresponding *f(x)* value lies in the open rectangle defined by the vertical lines and *,* and the horizontal lines and . Alternatively, if we take here. Then for any given , it will be possible to find so that if the values are less than and are within units of (but not equal to ), then the corresponding values will lie within units of . (Answers may vary)

1. For , suppose that we only wanted to consider values for that were greater than or we only wanted to consider what happens to as approaches from the right, if the values were within units of but not equal to , would the corresponding values be forced to lie within units of ?

**Answer:** Taking *L* = 3, for any value of , it is possible to find a value so that whenever *x* > 0 and then the corresponding *y* or *f(x)* value lies in the open rectangle defined by the vertical lines and *,* and the horizontal lines

 and . Alternatively, if we take ,then for any given , it will be possible to find so that if the values are greater than and are within units of (but not equal to ), then the corresponding values will lie within units of . (Answers may vary)

1. Take .
2. Find if possible, and state it here. **Answer:** *f(0)*does not exist.
3. Position the point labeled on the -axis at the location from part a.
4. As the values get closer to the value, do the corresponding or values get close to any fixed value on the -axis? If so, what do you think the limit of as approaches might be?

**Answer:** No, not applicable

1. Position the point labeled on the -axis at the location chosen in the preceding step.
2. For , if possible, find a respective value of so that if you pick any coordinate within units of point , then the corresponding or coordinate lie within units ofon the -axis.

**Answer:** Not possible

1. Do you think the exist? If so, what is it? **Answer:** No, not applicable
2. Explain why or how you know the limit exists or why the limit does not exist. That is, explain how the graph demonstrates or fails to demonstrate fulfillment of the definition of a limit.

**Answer:** In this case, limit does not exist. We know this because if the limit as *x* approaches 0 exists, it must lie in (-1, 1), but there are values for such that for any , there is at least one *x* value satisfying that has an *f(x)* value that does not lie in the open rectangle defined by the vertical lines and *,* and the horizontal lines

 and . Alternatively, there is at least one ε > 0 for which it is not possible to find a δ value so that if then . (Answers may vary)