Exploring Geometric Series

In this activity, we will explore the following question:

Is it possible to add infinitely many positive numbers and NOT get infinity as the sum?

We will discover what the following sums equal:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = ?$$

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots = ?$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots = ?$$

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+\cdots = ?$$

What do these formulas mean? Where do they come from? We will see!

To start, pick one of the above formulas and explain what you think it might mean:

Part 1: Hands-On Exploration with String

1. In **groups of three**, perform the following steps with a one-meter piece of string:

    - One student (Person A) cuts the string into thirds and gives one-third to each of the other two students (Persons B and C).
    - Person A then divides their remaining third into thirds again, giving each of these thirds to the other two students.
    - Continue this process, with Person A progressively dividing the remaining string into thirds and giving one of the thirds to each of Persons B and C.

2. Observe and record what happens after many iterations. What fraction of the string do Persons B and C each get closer and closer to having?

   ____

   Explain how this illustrates the following infinite series:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

3. Now, get in **groups of four**, and do the same thing, but with Person A dividing the string into fourths instead of thirds, and repeating the process.

   What fraction of the string do Persons B, C, and D each get closer and closer to having? _________

   What series does this illustrate? Write it below:

   ____

Each of these is an example of a "geometric series." A **geometric series** is a series in which each term after the first is found by multiplying the preceding term by a fixed, non-zero number called the **common ratio**.

Is the following a geometric series? YES / NO (circle one)

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+\cdots$$

Explain why or why not:

Write a geometric series of your own:

____

Part 2: Algebraic Exploration

1. Simplify the following expression and observe the result:

   $(1 - x)(1 + x + x^2 + x^3 + x^4) =$

2. Without expanding everything out, make a conjecture about what the following expression would simplify to. Then confirm your conjecture by expanding.

   $(1 - x)(1 + x + x^2 + x^3 + x^4 + x^5) =$

3. Generalize the result by extending the expression to:

   $(1 - x)(1 + x + x^2 + x^3 + \cdots + x^n) =$

4. Divide both sides of the above equation by $(1 - x)$. Write your simplified answer below:

5. Examine the behavior of the equation found in (4) as n "approaches infinity":

   - The left-hand side becomes the <u>infinite</u> series: $1 + x + x^2 + x^3 \ldots$
   - On the right-hand side, if $|x| < 1$ (i.e., if x is between -1 and 1), then the term x^{n+1} gets small ("approaches zero") as n gets large ("approaches infinity").

   We therefore obtain the following formula (assuming $|x| < 1$):

$$\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}}$$

6. Sometimes, we would prefer the left-hand side to begin with x (rather than beginning with 1). Subtract 1 from both sides of this equation and simplify the right-hand side (i.e., find a common denominator, etc.).

   Fill in the blank below. (Assume $|x| < 1$.)

$$x + x^2 + x^3 \ldots = \underline{\qquad\qquad\qquad}$$

7. Pick a value for value for x (with $|x| < 1$) and plug it into the above formula. Write out the resulting formula below and simplify the right-hand side.

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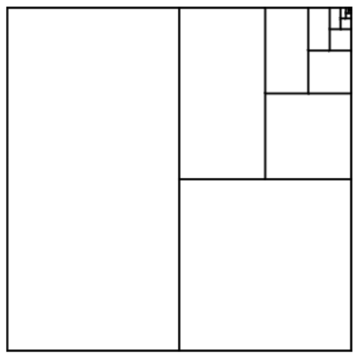
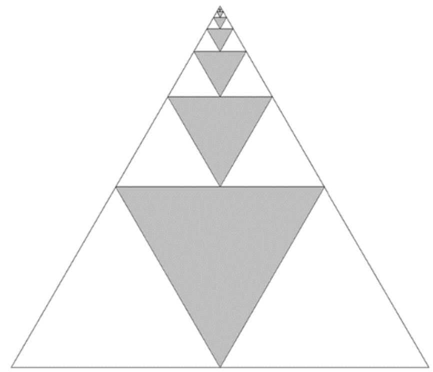
Part 3: Geometric Exploration





8. Explain how the left diagram (above) can be used to show

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

9. The second diagram also illustrates a geometric series. Write the series below:

____

[At this point, your instructor may provide some other examples on the board.]

10. Collaborate with fellow students to try to come up with a diagram of your own that illustrates a geometric series.

Diagram: Series:

Part 4: Repeating Decimals

Consider the repeating decimal: 0.111111 ...

What fraction does this represent? _______________

[Hint: Consider the decimal 0.333333 ... What does it equal as a fraction?]

You may use a calculator to check your answer.

Now, write out 0.111111 ... as an infinite series. [Hint: $0.1 + 0.01 =$?]

0.111111 ... =

What is the common ratio of this geometric series? __________

Plug this common ratio value in for x into the formula you found in question 6.

What do you notice?

Part 5: Real World Application (Bouncing Ball)

11. Consider the following scenario:

    > A ball is dropped from a height of 6 feet and bounces back to three-fourths of its previous height each time that it bounces. Calculate the total distance traveled by the ball after infinitely many bounces.

    You may collaborate with your fellow students.

BONUS QUESTION:

12. Think of a real-world situation where an infinite geometric series might apply. Describe the situation and formulate the series.

Bonus Homework Extension:

- Use a full sheet of paper and a ruler/straightedge to draw a (large) triangle, and label the vertices A, B, and C.

- Use the ruler to draw the three "medians" of the triangle (i.e., the line segments that connect the vertices to the midpoints of the opposite sides). If you have done this correctly, the three medians should intersect at a single point, called the "centroid" of the triangle. Label the centroid S_1.

- These three medians divide the triangle into 6 smaller triangles that have equal area. (Can you explain why these must have equal area?)

- Consider the median that has A as one of its endpoints. Label the other endpoint D (i.e., so that D is the midpoint of the segment BC.)

- Consider triangle ADC. We will progressively shade in triangle ADC as follows:

    - The triangle ADC is comprised of three smaller triangles. Use a colored pencil (or crayon, marker, etc.) to shade in two of them, the two that do NOT have CD as one of their sides.

    - Next, consider the triangle BCS_1, and draw the three medians for that triangle. This creates another 6 triangles. Label the centroid S_2. Use a different color to shade in the two of these 6 triangles that are contained inside ADC and do NOT have CD as one of their sides.

    - Repeat this process several times (until the triangles become too small to shade), labeling the subsequent centroids as $S_3, S_4, \ldots$ (each of these centroids should lie along the line AD), and always shading in two small triangles that are contained inside ADC and that do not have CD as one of their sides.

- What geometric series does this diagram illustrate? Explain your reasoning.