Exploring Geometric Series

In this activity, we will explore the following question:

Is it possible to add infinitely many positive numbers and NOT get infinity as the sum?

We will discover what the following sums equal:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=?$$

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=?$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=?$$

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+\cdots=?$$

What do these formulas mean? Where do they come from? We will see!

To start, pick one of the above formulas and explain what you think it might mean:

Answers will vary. Students may notice, for example, with the first series that when the terms are added successively using a calculator, they appear to be getting closer and closer to the number 1.

Part 1: Hands-On Exploration with String

1. In **groups of three**, perform the following steps with a one-meter piece of string:

    - One student (Person A) cuts the string into thirds and gives one-third to each of the other two students (Persons B and C).
    - Person A then divides their remaining third into thirds again, giving each of these thirds to the other two students.
    - Continue this process, with Person A progressively dividing the remaining string into thirds and giving one of the thirds to each of Persons B and C.

2. Observe and record what happens after many iterations. What fraction of the string do Persons B and C each get closer and closer to having?

$$\frac{1}{2}$$

Explain how this illustrates the following infinite series:

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

Originally, Persons B and C each receive a piece of string that has length 1/3. Then they receive a string that has length 1/3 of 1/3 (or 1/9). Then they receive a string that has length 1/3 of 1/3 of 1/3 (or 1/27). So, the total length of string that they receive is 1/3 + 1/9 + 1/27 +… But Persons B and C each get closer and closer to having 1/2 of the string (because they have the same amount as each other, and because Person A's remaining string is progressively getting smaller and smaller so that it approaches a length of 0.). Therefore: 1/3 + 1/9 + 1/27 +… = 1/2.

3. Now, get in **groups of four**, and do the same thing, but with Person A dividing the string into fourths instead of thirds, and repeating the process.

What fraction of the string do Persons B, C, and D each get closer and closer to having? $\frac{1}{3}$

What series does this illustrate? Write it below:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\cdots=\frac{1}{3}$$

Each of these is an example of a "geometric series." A **geometric series** is a series in which each term after the first is found by multiplying the preceding term by a fixed, non-zero number called the **common ratio**.

Is the following a geometric series? YES / NO (circle one)

$$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+\cdots$$

Explain why or why not:
It has a common ratio of 1/5 (i.e., each term is one-fifth of the previous term)

Write a geometric series of your own:

Answers will vary. One possible example (with common ratio 2/3) is:

$$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots$$

Part 2: Algebraic Exploration

1. Simplify the following expression and observe the result:

$$(1-x)(1+x+x^2+x^3+x^4) = 1+x+x^2+x^3+x^4-x-x^2-x^3-x^4-x^5$$
$$= 1-x^5$$

Notice the "telescoping" nature of the expression (i.e., most of the terms "cancel").

2. Without expanding everything out, make a conjecture about what the following expression would simplify to. Then confirm your conjecture by expanding.

$(1-x)(1+x+x^2+x^3+x^4+x^5) = 1-x^6$ (conjecture)

$$(1-x)(1+x+x^2+x^3+x^4+x^5)$$
$$= 1+x+x^2+x^3+x^4-x-x^2-x^3-x^4-x^5-x^6$$
$$= 1-x^6 \quad \text{(confirmation)}$$

3. Generalize the result by extending the expression to:

$$(1-x)(1+x+x^2+x^3+\cdots+x^n) = 1-x^{n+1}$$

4. Divide both sides of the above equation by $(1-x)$. Write your simplified answer below:

Work:

$$\frac{(1-x)(1+x+x^2+x^3+\cdots+x^n)}{1-x} = \frac{1-x^{n+1}}{1-x}$$

Final Answer:

$$1+x+x^2+x^3+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$$

5. Examine the behavior of the equation found in (4) as n "approaches infinity":

   - The left-hand side becomes the <u>infinite</u> series: $1 + x + x^2 + x^3 \dots$
   - On the right-hand side, if $|x| < 1$ (i.e., if x is between -1 and 1), then the term x^{n+1} gets small ("approaches zero") as n gets large ("approaches infinity").

   We therefore obtain the following formula (assuming $|x| < 1$):

$$\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}}$$

6. Sometimes, we would prefer the left-hand side to begin with x (rather than beginning with 1). Subtract 1 from both sides of this equation and simplify the right-hand side (i.e., find a common denominator, etc.).

   Original Formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

   Subtract 1 from both sides:

$$x + x^2 + x^3 + \cdots = \frac{1}{1-x} - 1$$

   Re-write to get a common denominator:

$$x + x^2 + x^3 + \cdots = \frac{1}{1-x} - \frac{1-x}{1-x}$$

   Simplify the right-hand side:

$$x + x^2 + x^3 + \cdots = \frac{1-(1-x)}{1-x}$$

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

   Fill in the blank below. (Assume $|x| < 1$.)

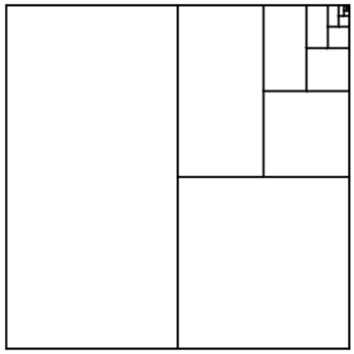
$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

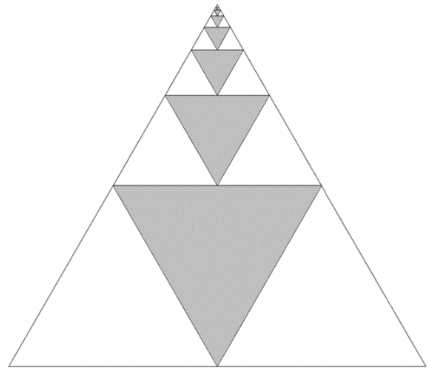
7. Pick a value for value for x (with $|x| < 1$) and plug it into the above formula. Write out the resulting formula below and simplify the right-hand side.

   Answers will vary. For example, if we let $x = \frac{1}{5}$, we get:

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}$$

Part 3: Geometric Exploration





8. Explain how the left diagram (above) can be used to show

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

Suppose the square has area 1. If we shade in 1/2 of the box (the left half of the square in the figure), then shade in an additional 1/4 (the lower right corner), then 1/8, and keep continuing in this manner, then we will have shaded in a total area of 1/2 + 1/4 + 1/8 +1/16 +… But the square gets progressively more filled-in so that after infinitely many iterations, the entire square is filled in. This shows that: 1/2 + 1/4 + 1/8 + 1/16 +… 1. (This can be confirmed by plugging $x=\frac{1}{2}$ into the equation found in question 6.)

9. The second diagram also illustrates a geometric series. Write the series below:

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}$$

[At this point, your instructor may provide some other examples on the board.]

10. Collaborate with fellow students to try to come up with a diagram of your own that illustrates a geometric series.

Diagram: Series:

Answers will vary here. The YouTube video may be used for inspiration.

Part 4: Repeating Decimals

Consider the repeating decimal: 0.111111 ...

What fraction does this represent? $\frac{1}{9}$

[Hint: Consider the decimal 0.333333 ... What does it equal as a fraction?]

You may use a calculator to check your answer.

Now, write out 0.111111 ... as an infinite series. [Hint: $0.1 + 0.01 = ?$]

$$0.111111\ldots = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots$$

$$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$$

What is the common ratio of this geometric series? $\frac{1}{10}$

Plug this common ratio value in for x into the formula you found in question 6.

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$$

What do you notice?

The answer from the formula (from question 6) is the same as the answer we expected from knowing that 1/9 is equal to 0.111111…

Part 5: Real World Application (Bouncing Ball)

11. Consider the following scenario:

    A ball is dropped from a height of 6 feet and bounces back to three-fourths of its previous height each time that it bounces. Calculate the total distance traveled by the ball after infinitely many bounces.

    You may collaborate with your fellow students.

    The ball initially drops 6 feet. Then it bounces back up to (3/4)6 feet. Then it falls back down that same amount. Then it bounces up to (3/4)(3/4)6 feet. Then it falls back down that same amount. Then it bounces back up to (3/4)(3/4)(3/4)6 feet. Etc.

    So, the total distance traveled by the ball is:

$$6 + 6\left(\frac{3}{4}\right) + 6\left(\frac{3}{4}\right) + 6\left(\frac{3}{4}\right)^2 + 6\left(\frac{3}{4}\right)^2 + 6\left(\frac{3}{4}\right)^3 + 6\left(\frac{3}{4}\right)^3 + \cdots$$

    which equals

$$6 + 12\left(\frac{3}{4}\right) + 12\left(\frac{3}{4}\right)^2 + 12\left(\frac{3}{4}\right)^3 + \cdots = 6 + 12\left(\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right)$$

$$= 6 + 12\left(\frac{\frac{3}{4}}{1 - \frac{3}{4}}\right)$$

$$= 6 + (12 \times 3)$$

$$= 42$$

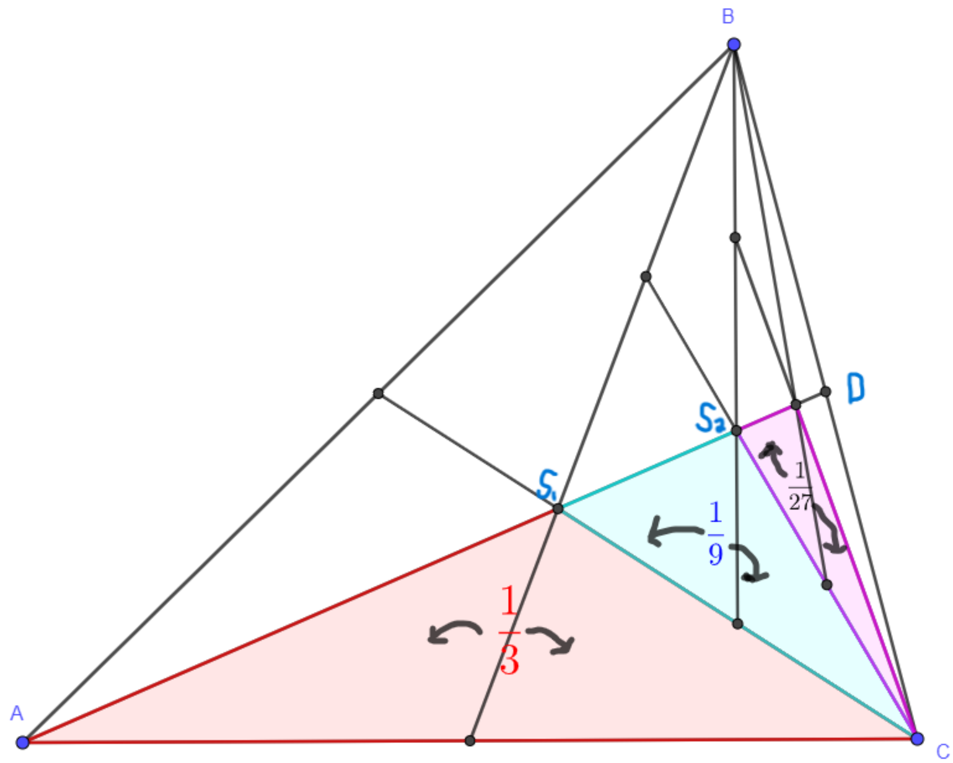
    So, the ball travels a total of <u>42 feet</u>.

BONUS QUESTION:

12. Think of a real-world situation where an infinite geometric series might apply. Describe the situation and formulate the series.

    Answers may vary. One interesting fact (from physics) is that the <u>time</u> between bounces (for the bouncing ball) forms a geometric series! So, the ball can bounce infinitely many times in a finite amount of time! If any students are interested in physics, this would be a great problem to work out to find the total time that the ball (in problem 11) is bouncing.

Bonus Homework Extension:



- Use a full sheet of paper and a ruler/straightedge to draw a (large) triangle, and label the vertices A, B, and C.
- Use the ruler to draw the three "medians" of the triangle (i.e., the line segments that connect the vertices to the midpoints of the opposite sides). If you have done this correctly, the three medians should intersect at a single point, called the "centroid" of the triangle. Label the centroid S_1.
- These three medians divide the triangle into 6 smaller triangles that have equal area. (Can you explain why these must have equal area?)
- Consider the median that has A as one of its endpoints. Label the other endpoint D (i.e., so that D is the midpoint of the segment BC.)
- Consider triangle ADC. We will progressively shade in triangle ADC as follows:
  - The triangle ADC is comprised of three smaller triangles. Use a colored pencil (or crayon, marker, etc.) to shade in two of them, the two that do NOT have CD as one of their sides.
  - Next, consider the triangle BCS_1, and draw the three medians for that triangle. This creates another 6 triangles. Label the centroid S_2. Use a different color to shade in

the two of these 6 triangles that are contained inside ADC and do NOT have CD as one of their sides.

- Repeat this process several times (until the triangles become too small to shade), labeling the subsequent centroids as $S_3, S_4, \ldots$ (each of these centroids should lie along the line AD), and always shading in two small triangles that are contained inside ADC and that do not have CD as one of their sides.

- What geometric series does this diagram illustrate? Explain your reasoning.

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$$

Suppose the area of the original triangle is 1. Then the first two triangles that are shaded have a combined area of 1/3 (i.e., each has an area of 1/6 and there are two of them). The next two triangles have a combined area of 1/3 of 1/3 (or 1/9). The next two triangles that are shaded have a combined area of 1/3 of 1/3 of 1/3 (or 1/27). So, the total area of all the triangles that get shaded after infinitely many iterations is 1/3 + 1/9 + 1/27 + 1/81 +… But these triangles progressively shade in the triangle ADC, which has an area of 1/2.