Name: ______________________________

Instructional Activity: The Rule of 70

Continuously Compounded Interest:

Recall that if P dollars is invested in an account that earns an annual interest rate r (expressed as a decimal), compounded continuously, then the amount of money A in the account after t years is given by:

$$A(t) = Pe^{rt}$$

We will investigate the following question in two ways: (1) graphically (using Desmos), and (2) algebraically.

<u>Initial Question:</u> Suppose \$10,000 is deposited in a bank account that earns 5% annual interest, compounded continuously. How long will it take for the money to double?

Desmos Activity:

Go to the following Desmos link and then answer the questions (below):

https://www.desmos.com/calculator/owszw2elw1

1. There are two ordered pairs highlighted in the Desmos activity, one in red and one in black.

   (a) What does the y-coordinate of the red point represent?

   The initial amount of money.

   (b) What does the x-coordinate of the red point represent?

   The initial time (time $t = 0$).

   (c) What does the y-coordinate of the black point represent?

   The doubled amount of money.

   (d) What does the x-coordinate of the black point represent?

   The time at which the money has doubled.

   (e) What does the numerical difference between the x-coordinates of the two points represent in practical terms?

   The amount of time it takes for the money to double.

2. When moving the slider for r (the interest rate), does the doubling time change? What about when moving the slider for P (the initial amount of money in the account)? Explain.

   The doubling time changes when moving the slider for r. This makes sense because as the interest rate increases, we would expect the investment to grow faster, and so have a shorter doubling time. But the doubling time does not change when moving the slider for P. The initial amount of money does not affect the doubling time.

3. Use the sliders in the Desmos activity to fill in the table below. Round to the nearest year. In the bottom two rows, select your own values.

Interest Rate	Doubling time (years)
2%	35
5%	14
7%	10
10%	7
14%	5
6%	12
8%	9

What patterns do you notice in the table?

Possible answers:

When the interest rate is multiplied by the doubling time, the result is 70 (or approximately 70).

When the interest rate goes up, the doubling time goes down.

The interest rate and doubling time are inversely proportional. For example, an interest rate of 10% is twice as much as an interest rate of 5%, whereas the doubling time is 1/2 as much (7 years vs. 14 years).

Predict the doubling time for 1% interest.

70 years (approximately)

Solving Algebraically:

4. Now solve the Initial Question algebraically. That is, solve:

$$20{,}000 = 10{,}000e^{.05t}$$

$$2 = e^{.05t}$$

$$\ln(2) = .05t$$

$$t = \frac{\ln(2)}{.05} \approx 13.86$$

13.86 years

5. Would your solution to #4 be the same if the amount of money deposited was \$15,000 (so that it doubled to \$30,000)? In other words, would the solution to $30{,}000 = 15{,}000e^{.05t}$ be the same as the solution to $20{,}000 = 10{,}000e^{.05t}$? Why, or why not?

   Yes, the solutions will be the same. In both cases, after dividing each side of the equation by the initial amount of money, we will be solving the equation $2 = e^{.05t}$. So, the initial amount of money does not affect the doubling time.

   Now go back to the Desmos activity to verify that your answer is correct.

   Using Desmos:

   Doubling time for \$10,000 at 5%: 13.863 years

   Doubling time for \$15,000 at 5%: 13.863 years

6. Use a calculator to find:

   $\ln(2) =$ .693147 $\qquad 100 \cdot \ln(2) =$ 69.3147

7. In problems #4 and #5, you should have found the solution to be $\frac{\ln(2)}{.05} = 13.86\ldots$ years. Now that we know that $100 \cdot \ln(2) \approx 70$, we can approximate $\frac{\ln(2)}{.05}$ without using a calculator:

$$\frac{\ln(2)}{.05} = \frac{100 \cdot \ln(2)}{100 \cdot .05} = \frac{69.3147\ldots}{5} \approx \frac{70}{5} = 14$$

   Notice that 14 years is a good approximation of 13.86 years!

   Now, suppose the interest rate is 7% (instead of 5%). Use the line of reasoning (above) to approximate the doubling time:

$$\frac{\ln(2)}{.07} = \frac{100 \cdot \ln(2)}{100 \cdot .07} = \frac{69.3147\ldots}{7} \approx \frac{70}{7} = 10$$

   Notice that the number 70 plays a prominent role (because $100\ln(2) = 69.3147\ldots$ which is approximately 70).

Fill in the blank below to formulate a "Rule of 70." Notice here that we are thinking of r as being a percentage rather than decimal (so an interest rate of 5% would give $r = 5$ rather than $r = .05$).

The Rule of 70:
If an amount of money grows at an annual interest rate of r%, compounded continuously, it will take roughly

____$70 \div r$____ *years to double.*

Questions to Assess Understanding:

- Suppose $30,000 is invested at an interest rate of 10%, compounded continuously. Approximately how many years will it take until the investment is worth $120,000?

  14 years. (At 10%, it will double roughly every 7 years. And the investment must double twice to get from $30,000 to $120,000.)

- For an investment of $100,000 at 7% annual interest, compounded continuously, which number most accurately describes how much the investment will be worth in 24 years? (a) $150,000 (b) $300,000 (c) $500,000 (d) $1,000,000 (e) $2,000,000. Explain your reasoning.

  (c) $500,000

  At 7% interest, it will double roughly every 10 years. So, after 24 years, it will double at least twice, but fewer than three times. So, the $100,000 investment will grow to something between $400,000 (doubling twice) and $800,000 (doubling three times).

- Our doubling-time rule is often called the "Rule of 72" instead of the "Rule of 70." But the number 70 is more accurate to use than 72 (because $100 \ln(2) = 69.3147$ ... is closer to 70 than to 72), so why would the number 72 sometimes be used? (Hint: What does the Rule of 72 say about doubling time for, say, 3%? What does the Rule of 70 say about doubling time for 3%?)

  72 is divisible by a lot of numbers (e.g., 1, 2, 3, 4, 6, 8, 9, 12, etc.). So, when determining the doubling time for, say 3% interest, the rule of 72 would say 24 years (a nice whole number), whereas the rule of 70 would say 23.333 years (a decimal).

- Suppose we want to come up with a rule for "tripling time." This rule could be called "The Rule of _110_" (fill in the blank with a number). Hint: What is ln(3)?

  $\ln(3) = 1.0986$ $\qquad\qquad$ $100 \cdot \ln(3) = 109.86 \approx 110$