TITLE OF LESSON: Rational Functions and Asymptotes

ESTIMATED TIME FOR LESSON (IN MINUTES): 75 minutes

SUGGESTED FORMAT (check all that are appropriate):

Individual in-class
Collaborative in-class
Individual homework
Collaborative homework

OVERVIEW:

This lesson focuses on rational functions formed by dividing a constant or a linear function by a product of linear factors (each of which differs from the numerator). Through guided explorations students will understand the relationships between factors of the numerator and denominator of a rational function and the function's zeros and its domain (including vertical asymptotes) both algebraically and graphically.

PREREQUISITE IDEAS AND SKILLS:

- Definition of rational numbers
- Intervals and interval notation
- Definitions of function, the domain of a function, the range of a function
- Graphing functions on Cartesian coordinate systems
- Algebraic and graphical representations of constant and linear functions
- Linear factors and zeros of a polynomial

MATERIALS NEEDED TO CARRY OUT LESSON:

- Access to Internet to show the Desmos graphs and some Geogebra applets given in the links inserted below
- Rational Functions Worksheet
- Rational Functions Answer Key

CONCEPTS TO BE LEARNED/APPLIED:

- How to divide a constant value or a linear function by a single linear function or a product of linear functions (that differ from the numerator) to form rational functions.
- How linear factors of the denominator of a rational function relate to the vertical asymptotes and domain of the function's graph.
- How the horizontal asymptote of a rational function is formed by considering the far end behavior of the rational function when its inputs approach positive and negative infinity.

INSTRUCTIONAL PLAN:

Day 1: Setting the stage (this can be done in the first ~15 minutes of class or the last ~15 minutes of the previous class).

Briefly remind students that rational numbers are defined as having the form a/b where a and b are integers but $b \neq 0$. Ask them to provide some examples.

Have students solve the following in pairs:

Task 1. Given $y_1 = 1$ and $y_2 = x - 2$,

a) find the quotient of the two, $y_3 = (y_1)/(y_2)$
   - algebraically
   - as points in a table that follows
b) Find the domain of y_3

Students should be quickly able to determine that $y_3 = 1/(x - 2)$ and then complete the following table.

x	y_1	y_2	$y_3 = (y_1)/(y_2)$
-3			
-2			
-1			
0			
1			
2			
3			
4			
5			

Whole Class Discussion

Lead a discussion to have students talk about why there is no value for y_3 when $x = 2$. In other words, discuss why the domain does not include 2. Try to guide the discussion so that students recognize that when:

- $x > 2$ the function y_3 has positive values, since 1 is always positive and a number larger than 2 minus 2 is always positive. So, you have a positive/positive = positive.
- $x < 2$ the function y_3 has negative values, since 1 is always positive and a number smaller than 2 minus 2 is always negative (e.g., 1.5 - 2 equals -0.5). So, you have a positive/negative = negative.

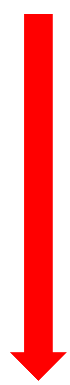
Task 2. Have students determine the range of $y_3 = 1/(x - 2)$

In small groups, have students think about more than just the sign of the outputs for the function y_3. Encourage them to consider the quantitative values of y by getting them to complete the following table. Have students do this using a calculator.

x	**y_3 = 1/(x - 2)**
1.5	
1.9	
1.99	
1.999	
2	
2.001	
2.01	
2.1	
2.5	

Make sure all students understand what happens as input values that are close to, but smaller than, 2, are used. It might help to show the graph, which can be found at https://www.desmos.com/calculator/w2bluowjb1 to trace it point by point to show that the output is approaching negative infinity. So, there is a vertical asymptote at x = 2.

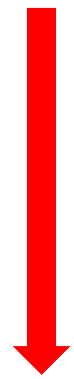
x	**y_3 = 1/(x - 2)**
1.5	1/(1.5 - 2) = 1/(-.5) = -2
1.9	1/(1.9 - 2) = 1/(-.1) = -10
1.99	1/(1.99 - 2) = 1/(-.01) = -100
1.999	1/(1.999 - 2) = 1/(-.001) = -1000
2	DNE (division by zero)



Input approaches negative infinity as x approaches 2 from the left

Also make sure students understand what happens when inputs larger than 2 get closer to 2 from the right. Again, it helps to show the graph, which can be found at https://www.desmos.com/calculator/w2bluowjb1 to trace it point by point to show that the output is approaching positive infinity.

x	y_3 = 1/(x - 2)
2.5	1/(2.5 - 2) = 1/(.5) = 2
2.1	1/(2.1 - 2) = 1/(.1) = 10
2.01	1/(2.01 - 2) = 1/(.01) = 100
2.001	1/(2.001 - 2) = 1/(.001) = 1000
2	DNE (division by zero)



Output approaches infinity as x approaches 2 from the right

Next, have students consider what is happening at the "far ends" of the graph by filling out this table.

x	y_3 = 1/(x - 2)
-1000	
-100	
-10	
10	
100	
1000	

Students should quickly see that as x approaches either negative or positive infinity, the output of the rational function approaches zero (as shown with the answers in red on the following table), which can be shown on the same graph as used earlier. This should lead to a discussion that y can be any positive or negative real number, but there is no way for it to be zero, since only a fraction with zero in the numerator can be zero. So, y = 0 is not in the range of the function, and there is a horizontal asymptote at y = 0.

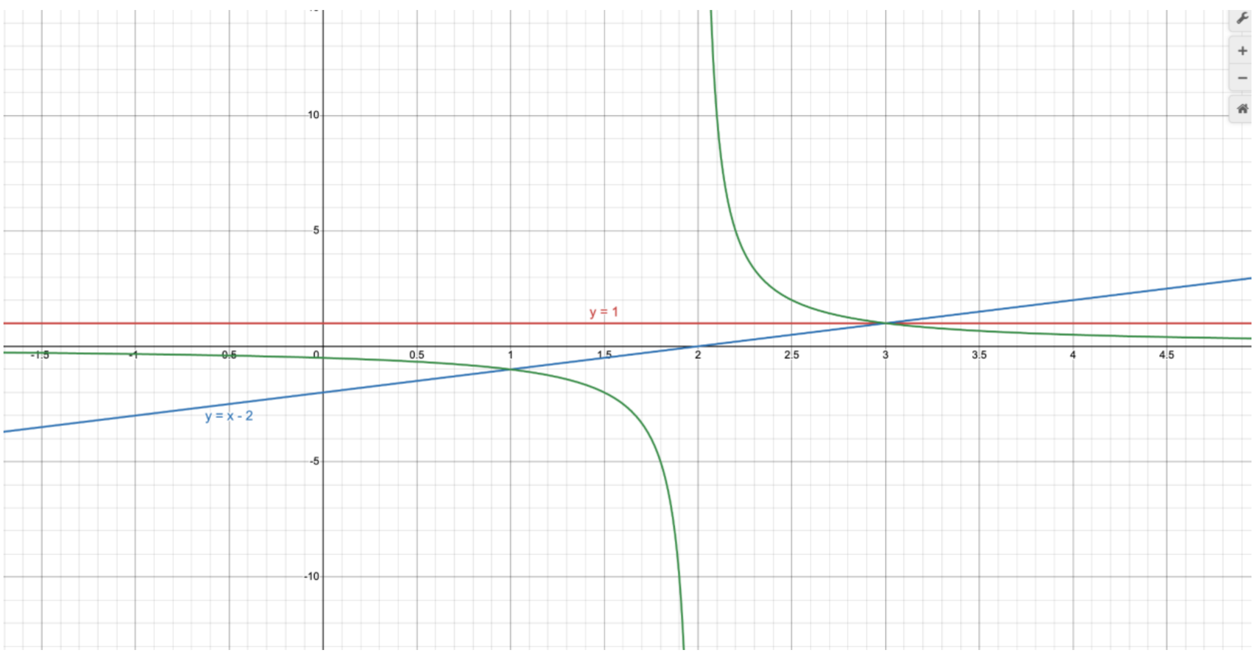
Output approaches zero as x approaches negative infinity

x	y_3 = 1/(x - 2)
-1000	1/(-1000 - 2) = 1/(-1002) ≈ -.00009998
-100	1/(-100 - 2) = 1/(-102) ≈ -.00980392
-10	1/(-10 - 2) = 1/(-12) ≈ -.08333333
10	1/(10 - 2) = 1/(8) = .125
100	1/(100 - 2) = 1/(98) ≈ .01020408
1000	1/(1000 - 2) = 1/(998) ≈ -.00100200

Output approaches zero as x approaches infinity

In-Class Discussion

Prompt the groups to discuss how what is happening in the table relates to the graphs of y_1, y_2, and y_3 in the following graph. Here is the Desmos file for the graph (where you can show or turn off each of the three graphs): https://www.desmos.com/calculator/nohmodjzna Be sure that students understand that y_3 is the green graph.



For example, students might notice that y_3 has two disconnected sections of the graph. They might even notice that each section is decreasing. Students will need to see that as the denominator (the blue function) approaches zero, then y_3 has different values depending if you approach from the left or from the right. Be sure that students understand that as the denominator of a rational function gets closer and closer to zero, then its output either must increase without bound if both the numerator and denominator are positive numbers, or it decreases without bound if both are negative.

Also, students will need to see that as the denominator (the blue function) increases more and more, then y_3 gets closer and closer (but never actually reaches) zero. The same occurs in the opposite direction.

In-Class Collaborative Worksheet
Next, have students work with this GeoGebra applet https://www.geogebra.org/m/Fsnt4mRk manipulating the h slider to create graphs to complete the first problem (with all its parts) on the Rational Functions and Vertical Asymptotes Worksheet.

Then have students use both the a and h sliders to create the graphs to complete the second problem (with all its parts) on the worksheet.

When students are drawing the graphs for the first and second problems on the worksheet, students should note why each has a vertical asymptote where it does.

In-Class Discussion
Let students discuss what they have noticed in creating the graphs above. Some points that should come up are:

- When the denominator of a rational function is zero, then there is no output. For the graphs that were shown, there was a vertical asymptote that corresponded to this.
- When the denominator of a rational function approaches zero, then its corresponding output value approaches either positive or negative infinity.
- The absolute value of the constant in the numerator of a rational function (where the function is y = a/(x - h) for some constants a and h) doesn't matter, but its sign does for the shape of the graph. However, neither the sign nor the absolute value of the constant in the numerator of a rational function impact the vertical asymptote of the function.
- If you have already covered translations, then students should note that
  - A function y = 1/(x - h) is the function y = 1/x that has been shifted h units horizontally.
  - A function y = a/(x - h) is the function y = 1/x that has been shifted h units horizontally AND stretched a units. It is also reflected about the x-axis by a factor of a.
  - Neither the stretch nor the reflection about the x-axis impact the position of the horizontal asymptote.

Out-of-Class Homework
Have students put (x - 1) in the denominator of the rational function on this site and turn on the button that shows horizontal and vertical asymptotes: https://www.geogebra.org/m/H6QcNpR
Then have them input the following linear factors for the numerator to see what graphs results recording the horizontal and vertical asymptotes for each when y =
3x, 3x + 1, 3x - 2, -2x, -2x + 3, -2x - 1, -x, -x + 2, -x - 3
Students should work on this exploration outside of class so that it can be discussed in the next class period.

Day 2: In-Class Discussion
Students should have noted, from doing their homework, that if a rational function with the form y = (ax - c)/(x - 1) has a vertical asymptote at x = 1 and a horizontal asymptote at a, regardless of the value of c.

In-Class Group Work
Break the students in small groups. Have students use the site (https://www.geogebra.org/m/H6QqcNpR) with the horizontal and vertical asymptotes on. Assign the following rational functions to each group to graph using the applet. The groups only need to mark the horizontal and vertical asymptotes for each function.

y = (x - 3)/(2x - 1)
y = (3x - 2)/(x - 1)
y = (4x + 3)/(3x - 1)
y = (2x + 4)/(4x - 3)

Circulate and point out to the groups that the functions each have a vertical and a horizontal asymptote. The vertical asymptotes should be relatively easy for the students, since they occur when the denominator is zero. However, you may need to prompt students to consider why the horizontal asymptote for each function occurs where it does.

Whole Class Discussion
Guide students to see that the very large and very small input values help provide information on the horizontal axis. Try to see if any students recognize that the horizontal asymptote for y = (ax + b)/(cx + d) would be y = a/c on their own. Try to lead the discussion to this by focusing on the fact that the b and d constant terms don't do much when the inputs are getting very large (that is approaching infinity) or when they are getting very negative (that is approaching negative infinity).

Note: This can be illustrated algebraically by multiplying numerator and denominator by 1/x (or by 1 over the highest power of x for a general rational function). In the resulting expression, terms with a power of x in the denominator clearly approach zero as x approaches positive or negative infinity, leaving the leading terms to dominate.

In-Class Group Work
Break the students into small groups. Have students use the site (https://www.geogebra.org/m/H6QqcNpR) with the horizontal and vertical asymptotes on. Have each group discuss the similarities and differences in the following three functions:

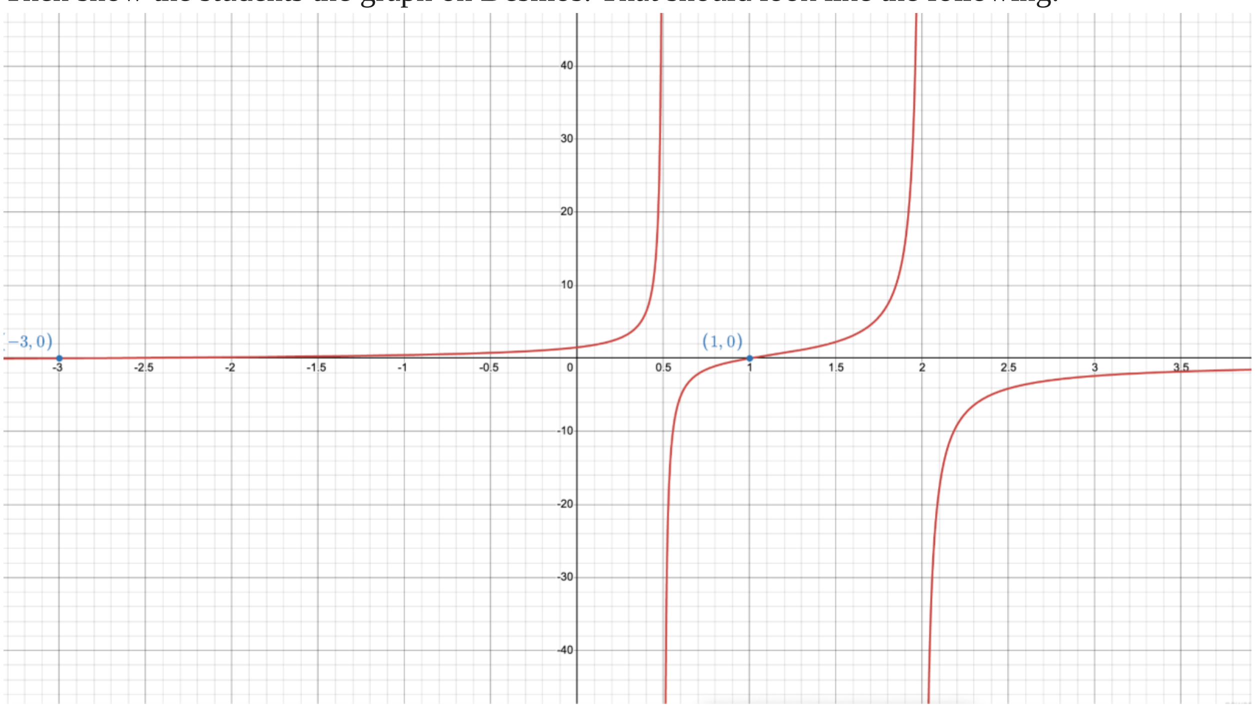
y = 4x/(3x - 1)(x - 1) and y = 4x/(3x - 1)(x -2) and y = 4x(x + 1)/(3x - 1)(x - 1)

Whole Class Discussion
Have students consider the following rational function, y = (x - 1)(x + 3)/(2 - x)(2x - 1). First, have them identify the vertical asymptotes, which occur at x = 1 and x = 2. Then have them

identify the zeros of the function, which occur at x = -3 and x = 1. Determine if the function is positive or negative over each region in the table. Use this information, along with the information on the vertical asymptotes and zeros of the function to have the students direct you how to sketch the graph of this function.

x	**y = (x - 1)(x + 3)/(1 - x)(2x - 1)**
x < -3	
-3 < x < ½	
½ < x < 1	
1 < x < 2	
x > 2	

Then show the students the graph on Desmos. That should look like the following.



Homework
Have students come up with four different rational functions that they put in this applet: https://www.geogebra.org/m/rmnvfqze They should draw the graph for each highlighting any asymptotes as well as writing the domain and range for each. (Note that this homework builds on what they have completed in this lesson, but it also sets the stage for the next lesson, especially if

a student has selected values on the slides that render a common linear factor in the numerator and denominator of the rational function.)

REFERENCES (FROM EXTERNAL RESOURCES)

All references used are listed as hyperlinks.