TITLE OF LESSON: Combining Linear Functions - Polynomials from Sums and Products)

ESTIMATED TIME FOR LESSON (IN MINUTES): 50-75 minutes

SUGGESTED FORMAT (check all that are appropriate):

Individual in-class
Collaborative in-class
Individual homework
Collaborative homework

OVERVIEW:

Through guided explorations, students will understand the relationships between linear factors of a polynomial function and the x-intercepts of its graph.

PREREQUISITE IDEAS AND SKILLS:

- Definition of a function
- Graphing functions on Cartesian coordinate systems
- Algebraic and graphical representations of constant and linear functions

MATERIALS NEEDED TO CARRY OUT LESSON:

- Access to Internet to show the Desmos graphs given in the links inserted below
- Zeros of Polynomials Worksheet
- Zeros of Polynomials Worksheet Answer Key

CONCEPTS TO BE LEARNED/APPLIED:

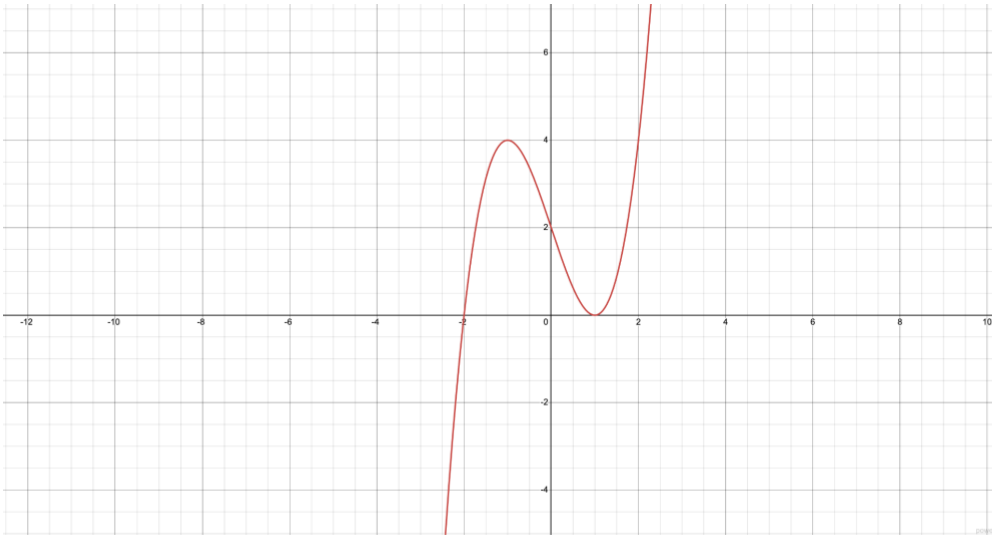
- How to add constant and linear functions
- How to multiply constant and linear functions
- That the linear factors of a polynomial equation are related to the zeros in the graph of that same polynomial

INSTRUCTIONAL PLAN:

Setting the stage (this can be done in the first 15-20 minutes of class or in the last 15-20 minutes of the previous class.

Briefly remind students of the definition of a function and that it can be represented many ways including as an equation, a table, or a graph. Also remind students that polynomials are a certain type of function that have the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_1 x + a_0$ where a_n, a_{n-1}, a_{n-2},..., and a_0 are real numbers and n is an integer.

As a motivator, show students the graph of $P(x) = x^3 - 3x + 2$ (see below) and ask them to notice the x-intercepts (or zeros), the values of x where $P(x) = 0$. Discuss how these are different types and tell them that this lesson will help them discover the difference in these types of zeros.



Have students solve the following in pairs:

Task 1. Given $y_1 = 2$ and $y_2 = 0.5x$, algebraically find:

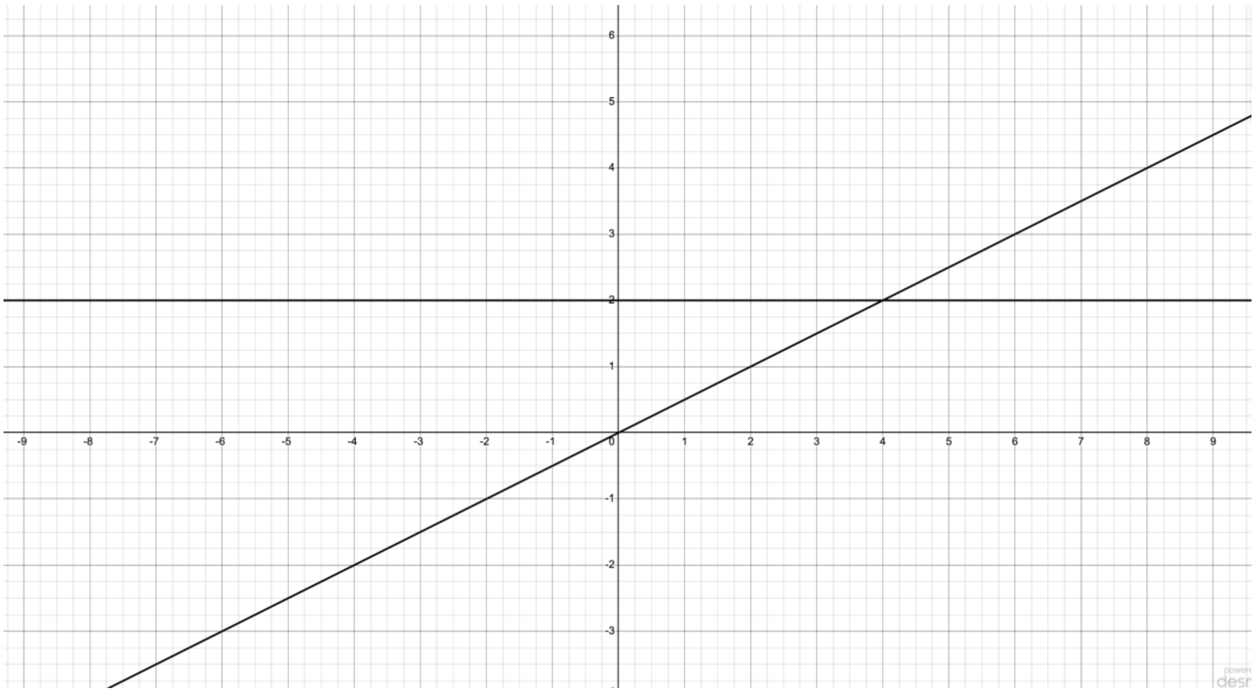
a) the sum of the two functions, y_3.

b) the product of the two functions, y_4.

Students should be quickly able to determine that:

- $y_3 = y_1 + y_2 = 2 + 0.5x$ (which some may write as $y_3 = 0.5x + 2$ or $y_3 = \frac{1}{2}x + 2$) for part a of the task
- $y_4 = (y_1)(y_2) = 2(0.5x) = x$ for part b of the task

Now show them the graphs of both functions y_1 and y_2 on the same set of axes (as shown using Desmos below) and have them solve the following collaboratively to justify their algebraic computations for y_3.

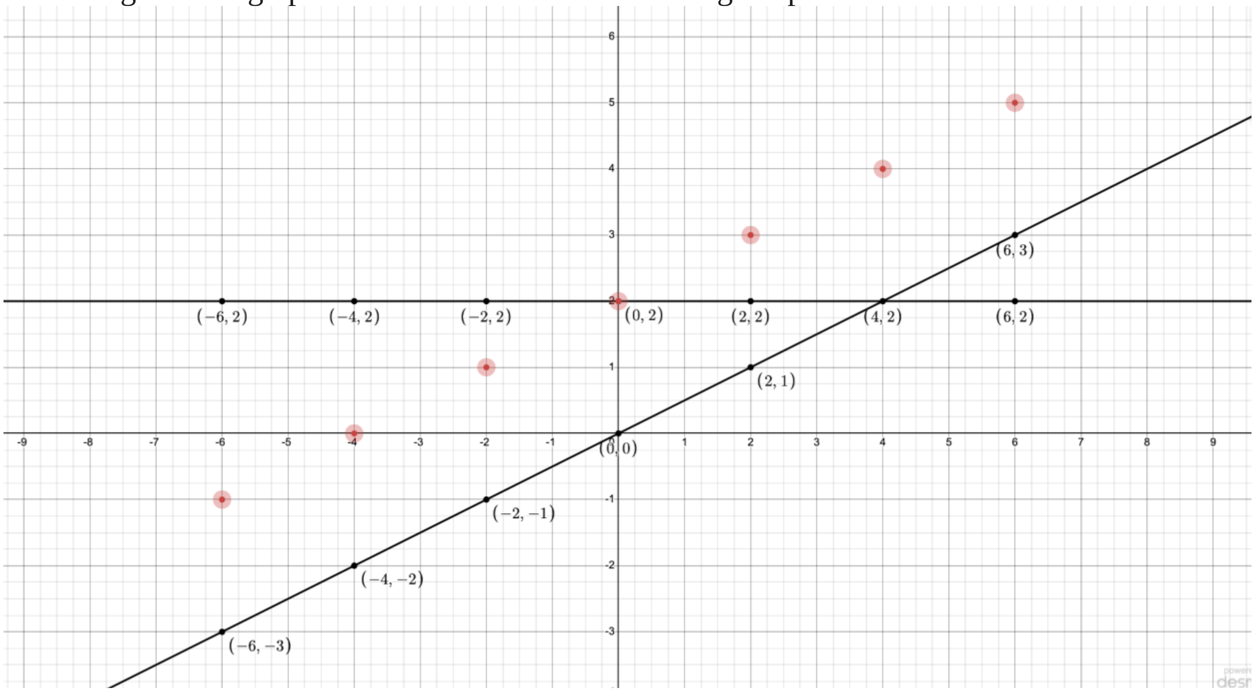


Suggest that they do so by working with pairs of points that have the same input value.

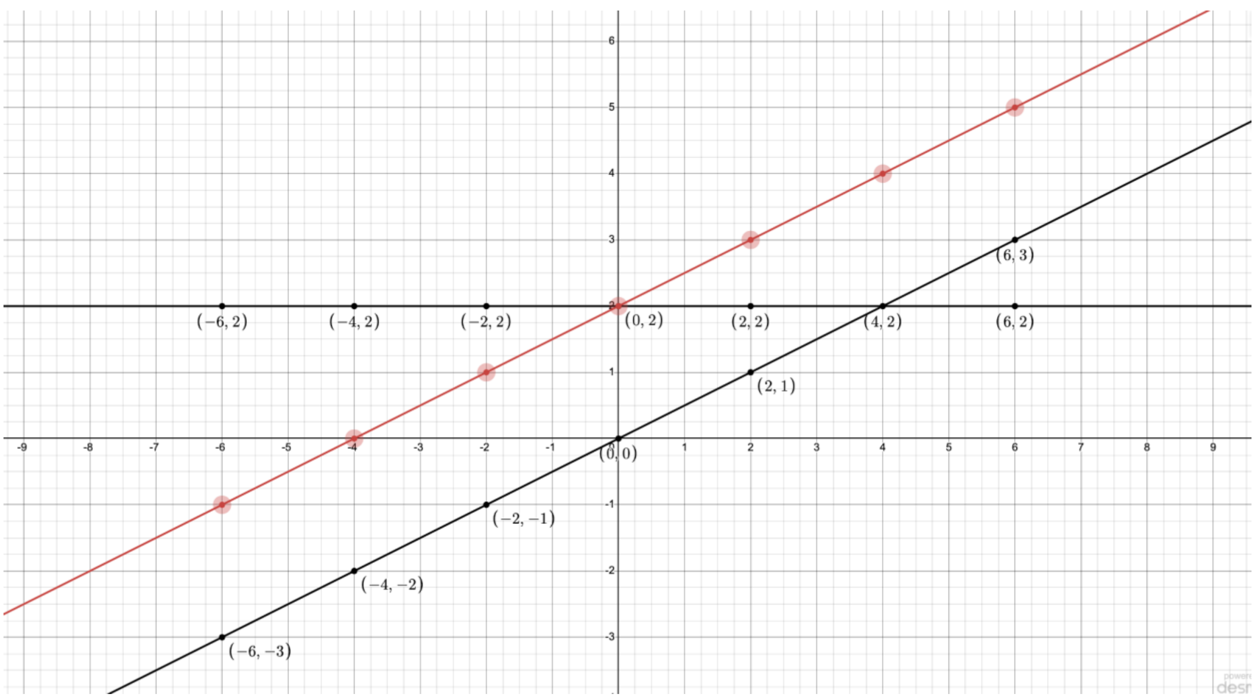
Task 2. Given $y_1 = 2$ and $y_2 = 0.5x$, graphically find the sum of the two functions, y_3. Encourage them to use a table, like the one that follows.

x	y_1	y_2	$y_3 = y_1 + y_2$
-6			
-4			
-2			
0			
2			
4			
6			

Prompt students to first consider both functions at x = 6, namely the points (6, 2) and (6, 3) to determine the resulting point when their output values are added, (6, 2+3) or (6, 5). It is helpful to mark the resulting point in a different color. Then have them repeat this process for corresponding points on the graphs of y_1 and y_2 to determine other resulting points for y_3. Be sure that students consider points with both negative and positive inputs. They should get something like the graph shown below when considering the points.



This should lead to the following graph (shown in red) for y_3. Here is the Desmos file, in case you want to use it: https://www.desmos.com/calculator/solwyhuajy For example, you can turn the labels on for the points that are marked for y_3.



Bring the class together for a brief discussion of this task. Try to make sure students explain how they completed the task. Here are good discussion points.

- When y = 0 in a point in one of the original functions, it acts as an additive identity.
- When both output values for two points with a shared x for y_1 and y_2 are positive, then y_3 is positive (e.g., consider points in y_1 and y_2 when x = 2) (When both output values for two points with a shared x for y_1 and y_2 are negative, but that doesn't occur in this example.).
- When y_1 and y_2 for two points with a shared x have opposite signs, then y_3 can be positive, zero, or negative depending on the outputs (e.g., consider values for y_1 and y_2 when x = -6, -4, -2)
- *Optional*: What has resulted is a vertical translation of 2 of the y_2 function, since y_1 is a constant. If you have already covered translations, you might want to explicitly point out to students that adding constant factors are the same as vertical translations, if they don't notice it themselves.

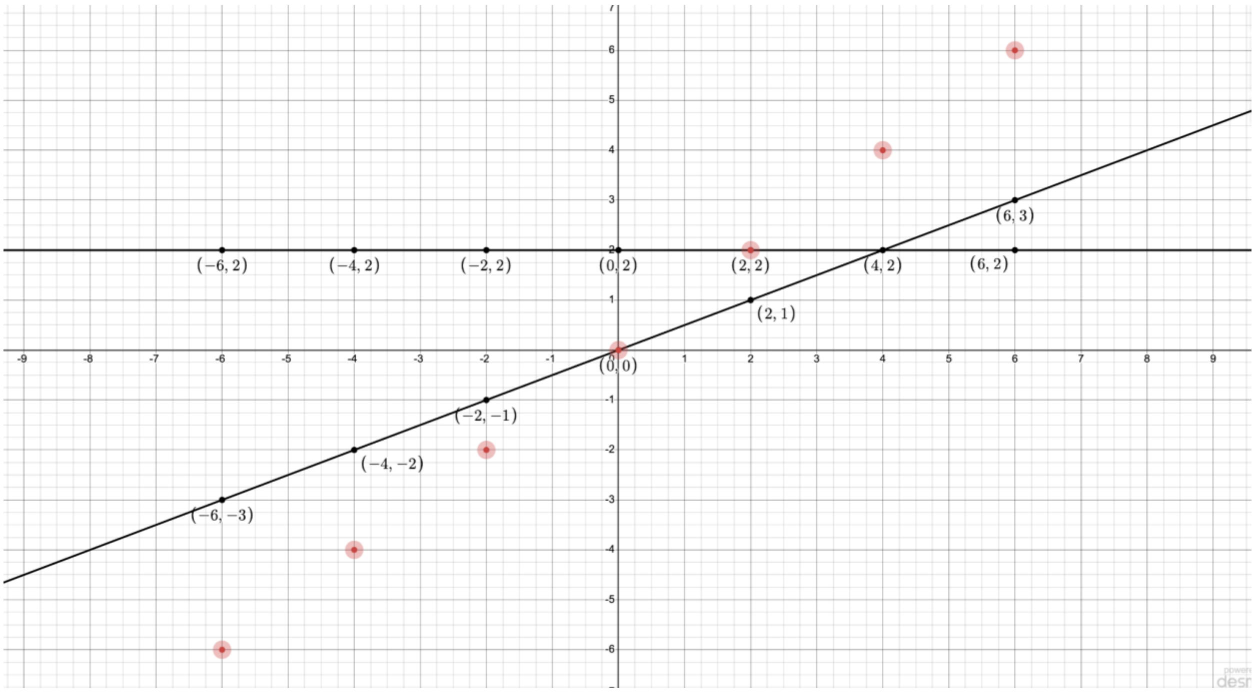
Now ask students to solve the following using the same point-by-point procedure.

Task 3. Given $y_1 = 2$ and $y_2 = 0.5x$, <u>graphically</u> find the <u>product</u> of the two functions, y_4.

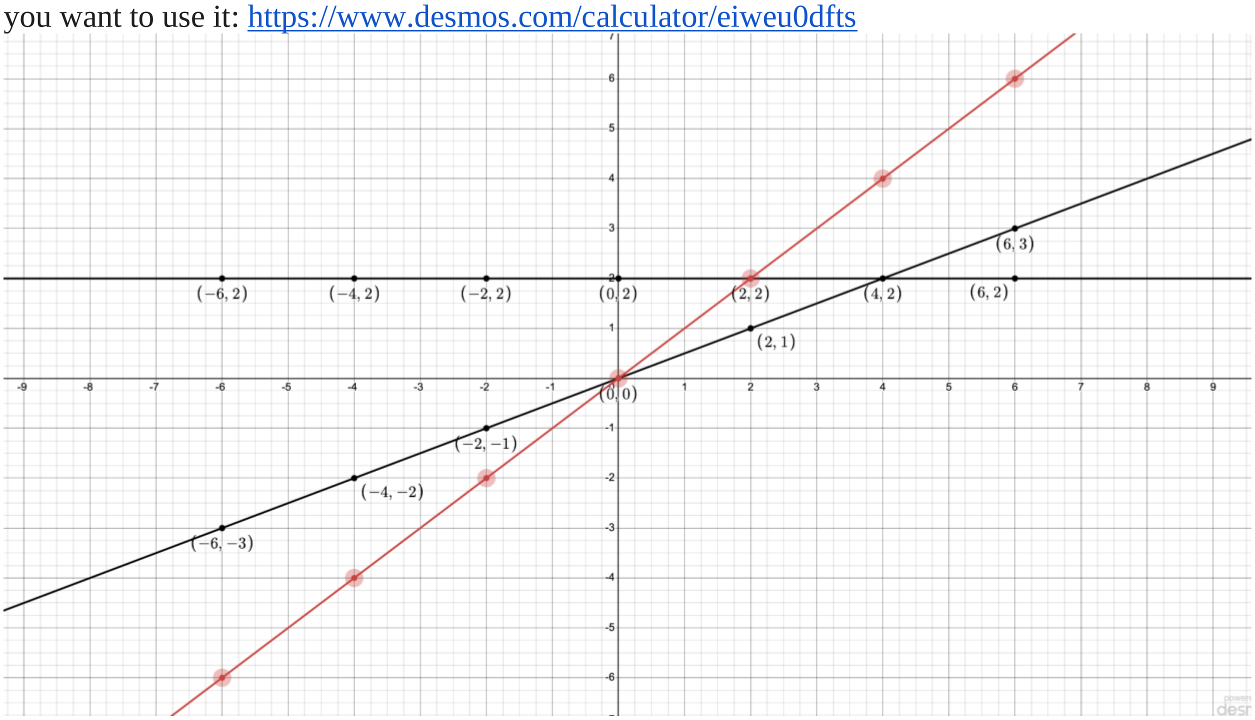
Encourage them to use a table, like the one that follows.

x	y_1	y_2	$y_3 = y_1 * y_2$
-6			
-4			
-2			
0			
2			
4			
6			

When students multiply the output values at different points, they should have a graph that looks like the following.



This should lead to the following graph (shown in red) for y_4. Here is the Desmos file, in case you want to use it: https://www.desmos.com/calculator/eiweu0dfts



Bring the class together for a brief discussion of this. Ask them which points helped them determine the graph? Try to draw out discussion so they explain how they completed the task. Here are good discussion points.

- When the y-value of a point in one of the original functions is 0, then the resulting product is 0. So, that point for y_3 has an output of 0.

- When both output values for two points with a shared x for y_1 and y_2 are positive, then y_4 is positive. (When both output values for two points with a shared x for y_1 and y_2 are negative, then y_4 would be positive, but this doesn't occur in this example.)
- When one of the output values for the two points with a shared x for y_1 and y_2 is positive and the other negative, then y_4 is negative.
- *Optional*: What has resulted is a stretch by a factor of 2 of the y_2 function, since y_1 is a constant. If you have already covered stretch transformations, you might want to point this out to students that multiplying by a constant factor results in a stretch, if they don't notice it themselves.

Now, have the students work on the following in small groups. This could take place in the next class or could extend on, depending on when you chose to do what is above.

Task 4: Given $y_1 = x + 1$ and $y_2 = x + 3$,

a) algebraically find the new function formed by the sum of the two functions y_1 and y_2; call it y_3
b) algebraically find the new function formed by the product of the two functions y_1 and y_2; call it y_4
c) geometrically (using graphs) find the sum function, y_3
d) geometrically (using graphs) find the product function, y_4

Here are the solutions.

a) $y_3 = (x + 1) + (x + 3) = 2x + 4$
b) $y_4 = (x + 1)(x + 3) = x^2 + 3x + 4$
c) Here is the solution graph for the sum graph y_3: https://www.desmos.com/calculator/4o6dskixru
d) Here is the solution graph for the product graph y_3: https://www.desmos.com/calculator/qpcnmicep2

Ask the students how the results of Task 4 differed from the previous results in Tasks 1-3. Be sure to get the discussion to cover the following points.

- The sum functions for Task 1a and Task 4a were both linear. When you add a constant and a linear function, the result is a linear function. When you add two linear functions, the result is a linear function.
- The product graphs differed (in Task 1b and Task 4b). When you multiply a constant and a linear function, the result is a linear function. When you multiply two linear functions, the result is a quadratic function.
  - Discuss why this is true.
  - Discuss what occurs in the equation when the y-value on the graph of a product function equals zero.
  - Extend the discussion to ponder what happens if you multiply three linear functions to form a third function. Where would this function's graph have zeros?

Task 5. Break the students into groups, where each group is asked to find the following product functions and their zeros (the input values that result in output values of 0) using Desmos. You may need to inform them how to reset the viewing window. Use the Zeros of Polynomials Worksheet for this activity (with its problems listed below). Note that the worksheet's answers are also provided in the Zeros of Poynomials Answer Key.

1. Graph y = (x - 1)(x + 2)(x + 3). Note this function is the product of y_1 = x - 1, y_2 = x + 2, and y = x +3. What are the zeros for this function?
2. Graph y = -x(x + 1)(x - 4). Note this function is the product of: y_1 = -x, y_2 = x + 1, and y_3 = x - 4. What are the zeros for this function?
3. Graph y = 3x(x - 5)(x + 1)(x + 2). Note this function is the product of y_1 = 3x, y_2 = x - 5, y_3 = x + 1, and y_4 = x + 2. What are the zeros for this function?
4. Graph the function y = (x - 3)$(x + 1)^2$. Note this function is the product of y_1 = x, y_2 = x + 1, and y_3 = x +1. What are the zeros for this function? Note that one of the linear factors, (x + 1), is squared; how does its resulting zero differ from the zero associated with the linear factor (x - 3). *[Introduction to multiplicity of zeros]*
5. Compare the graph of y = (x - 2)(x+3) to each of the following graphs, one at a time. State how the graph of each is similar and how each is different.
   a. $y = (x - 2)^2(x + 3)$
   b. $y = (x - 2)(x + 3)^2$
   c. $y = (x - 2)^2(x + 3)^2$
   d. $y = (x - 2)^2(x + 3)^3$

Homework [answers will vary]:
Have students engage with the following Geogebra app using its sliders.
https://www.geogebra.org/m/CPvzAemM
The assignment is to:

1. Write out 5 functions (in terms of their linear factors).
2. State what the zeros are for each function.
3. Draw the graph of each function.

The students should be instructed that they should include at least two examples with a repeating linear factor, which is likely to be written with exponents. For example, if given the function y = (x + 2)(x + 2), it should be written as $y = (x + 2)^2$.

Extension Activity #1 (This is best done collaboratively in class and only takes about 10 minutes.)
Have groups of students engage with the following Geogebra app using its sliders to study the multiplicity of zeros: https://www.geogebra.org/m/ycJyWe5s

Ask the students to discuss what they note when a factor has an even exponent. Ask them to also describe what they notice when a factor has an odd exponent.

Bring the groups together for a whole class discussion to make sure they realize that a zero with even multiplicity acts locally as an even function and a zero with odd multiplicity acts locally as an odd function. (Students may be familiar with the terms “bounce” or “turn” for even behavior or a “cross” for odd behavior.)

Extension Activity #2:
[This doesn’t take very long since y = (x + 1)(x + 3) is one of the graphs above, but it is very helpful for students. It can be given as a homework problem, or it can be explored in the next class.]
Show that f(x) = (x + 1)(x + 3) also is equivalent to g(x) = $(x + 2)^2$ - 1, since both expand to be the equation y = x^2 + 3x + 4. Have students discuss the benefits of writing the equation in the format given by f(x) and the format given by g(x).

REFERENCES (FROM EXTERNAL RESOURCES)

All references used are listed as hyperlinks.