Exploring the Connection between Secant Lines and the Tangent Line at a Given Point

Introduction: Have you ever wondered how to find the slope of a graph at a certain point when the graph is not a line? Let's see if we can use our prior knowledge of computing the slope of a line to help us estimate and find the slope of a nonlinear graph at a given point.

Step 1: Consider the graph of $y = 5 - x^2$.

(a) Refer to the activity in Desmos at Desmos Activity. Using at least two different computations, estimate the slope of the graph at the point $P = (1, 4)$ and record your answer below.
Suggestion: Zoom in until the graph begins to look linear "near" the point P. Then, use the grid to help you estimate the slope. For the second computation, consider zooming in more and using different points to estimate the slope.

Estimated slope: ____

Hint: You can"zoom in" on the graph using the "+" button or the mouse scroller. And, you may also "click and drag" the screen to move it as needed. For information regarding the Graph Settings in Desmos, please refer to the short video found at: Desmos Graph Settings (1 minute).

(b) Consider the estimated slope you found in part (a). What does the value of your estimated slope convey about the variation of x and y? Be specific.

Step 2: Use a calculator and your prior knowledge regarding lines to complete the following table where you will:

(a) compute a corresponding y-value for a given x-value where $y = 5 - x^2$.

(b) find the slope of the line between the named point and the point $(1, 4)$.

Terminology: A line passing through two points of a curve is called a *secant line*. In this step, we will compute the slope for various secant lines "near" the point $(1, 4)$. That is, we will be finding various *average rates of change* "near" the point $(1, 4)$. We can then use these slopes (or average rates of change) to approximate the slope of the graph at the point $(1, 4)$.

Point Name	x-value	y-value	Slope of secant line between the named point and the point $(1, 4)$.
A	0.75		A and (1,4) : $m =$
B	0.80		B and (1,4) : $m =$
C	0.85		C and (1,4) : $m =$
D	0.90		D and (1,4) : $m =$
E	0.95		E and (1,4) : $m =$
F	1.05		F and (1,4) : $m =$
G	1.10		G and (1,4) : $m =$
H	1.15		H and (1,4) : $m =$
I	1.20		I and (1,4) : $m =$
J	1.25		J and (1,4) : $m =$

Step 3: Return to the Desmos activity.

(a) Return the graph to the default settings by clicking on the "home" button in the top-right corner.

(b) "Turn on" the next two tabs (labeled 8 and 14) by selecting the circle (radio button) on the left-hand side of the tab. Use the additional information that appears on the graph to check your answers in the chart on Step 2. "Click and drag" the moveable purple point on the graph as needed. Notice that the appropriate secant line appears as the purple point is moved.

(c) For more details on how the various slopes are computed, "turn on" the next tab (labeled 20).

Step 4:

(a) Using the information in the table from Step 2, make a prediction for the slope of the graph at $a = 1$.

Hint: As x gets closer to 1, do the corresponding slopes appear to approach a particular number? What is it? Record your answer below.

Prediction for the slope: ____

(b) How does this prediction compare to your estimate in Step 1? Does this make sense? Why or why not?

Step 5: Using $f(x) = 5 - x^2$, find the expression for $\frac{f(x)-f(a)}{x-a}$. Simplify your answer.

Step 6: Find the expression $\frac{f(x)-f(a)}{x-a}$ where $a = 1$ using your work in Step 5.

Step 7:

(a) Using calculus, compute $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ when $a = 1$.

Suggestion: Use your work from Step 6.

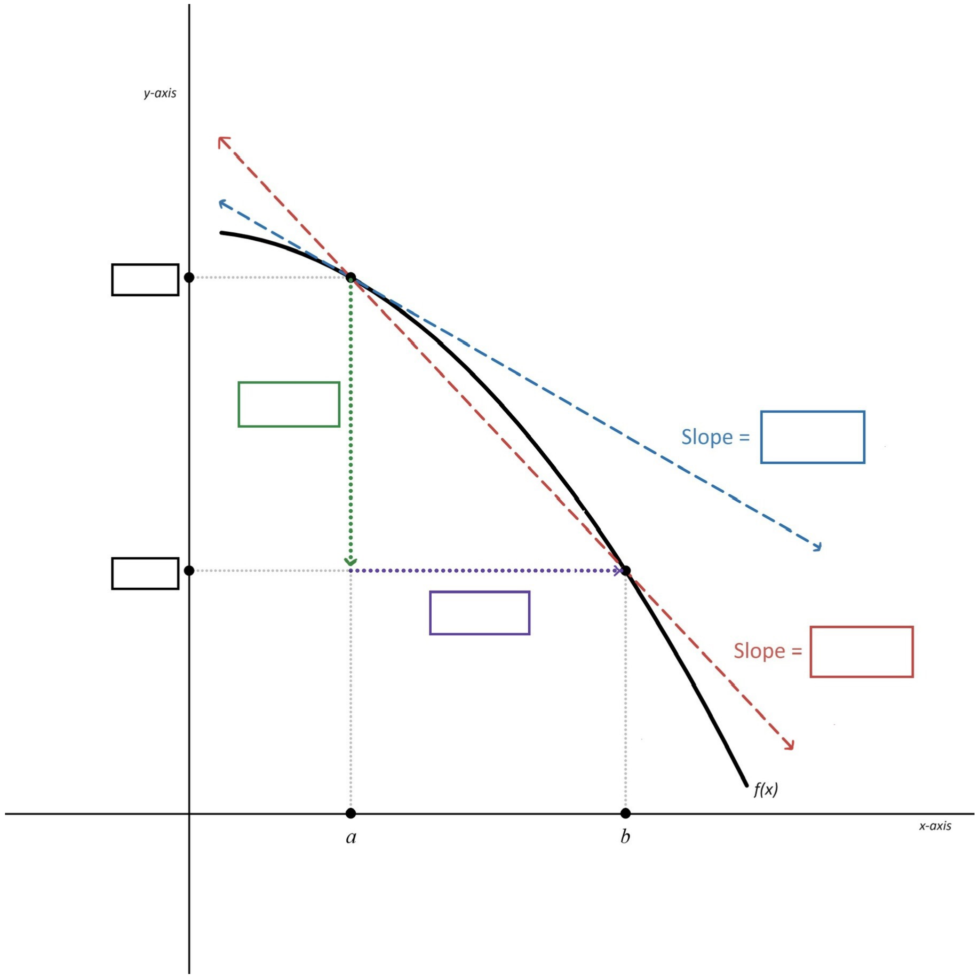
(b) Does this answer match your estimate for the slope of the graph at $a = 1$ in Step 1 and/or your prediction for the slope in Step 4? Does this surprise you? Why or why not?

(c) The tangent line to the graph at $a = 1$ is the line that goes through the point $P = (1, 4)$ and has the same slope as the graph at that point.

Find the equation of the tangent line of $f(x) = 5 - x^2$ at $a = 1$.

Terminology: The slope of the graph at the point $(1, 4)$ is the *instantaneous rate of change* at the point $(1, 4)$.

(d) Before continuing, let's generalize the quantities utilized in the difference quotient. Consider the below graph, $f(x)$, and the given secant line and tangent line. Write an expression in each of the six boxes that represents the respective graphical quantity.



Step 8: Return to the Desmos activity.

"Turn on" the tab labeled 38 and compare your equation of the tangent line to the one drawn.

Step 9: It's your turn to play in Desmos!

Before making changes in Desmos, it is highly recommended that you "turn off" tabs 8, 14, 20, and 38. To do so, deselect the radio buttons to the left of those tabs.

(a) In Desmos, tab 2 has a slider for the value of "a." Move the slider to a different value of "a." Predict the instantaneous rate of change of the function at $x = a$. Record your choice for "a" and your prediction below.

Value of a: ____________

Prediction for the instantaneous rate of change at $x = a$. ____________

(b) In Desmos, use tabs 8, 14, and 20, to graph the secant lines as the moveable purple point approaches your chosen point at $x = a$. Do the slopes of the secant lines (i.e. the average rates of change) approach your prediction for the instantaneous rate of change at $x = a$?

(c) Using calculus, find the instantaneous rate of change at $x = a$ by computing $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ for the a value you chose.

(d) Does this answer match your prediction for the instantaneous rate of change at your chosen a value? Why or why not?

(e) Find the equation of the tangent line of $f(x) = 5 - x^2$ at your chosen a value.

(f) Compare the equation of the tangent line to the graph of the tangent line given in Desmos ("turn on" tab 38 to view the tangent line). Does it match?

Step 10: Let's play some more in Desmos!

Again, it is recommended that you begin this step with tabs 8, 14, 20, and 38 turned off.

Create your own function and a value. Change the function in tab 1 to your new function and move the slider in tab 2 to your preferred a value.

Find the equation of the tangent line to your new function at your new a value. Feel free to use tabs 8, 14, 20, and 38 to verify your work.