Mean Value Theorem Worksheet

Instructor Notes

The following worksheet is designed to take the place of formal instruction with regard to the Mean Value Theorem. That is, the Mean Value Theorem should not be addressed prior to students working on this worksheet. A sample set of directions for students is as follows:

Directions: Complete the following problems/prompts. Submit your work/thought process and answers. You may use a graphing utility. Do not use any outside resources.

Notes:

- This activity is designed to come after curve sketching so that students have more calculus language/tools to answer some questions.
- The graphing utility could be optional. There are two problems within this worksheet in which students are to graph functions. Those problems could be modified to have students graph those functions using calculus. That is, they could be guided to graphing those functions by taking into consideration x and y-intercepts, asymptotes, critical points, inflection points, local extrema, increasing/decreasing intervals, and concavity. That is not done here because this is already a long activity, and I don't want that process taking away from the main theme of this activity.
- The intention of the worksheet is to have them learn about the Mean Value Theorem. The direction about not using outside resources is intended to force students to rely on information they already know in order to have a more in depth understanding of the Mean Value Theorem.
- This activity is not intended to be completed in class. Time is needed to process various aspects, so I don't like including a time constraint on it.
- This activity has been administered two different ways. One approach is to give it to them as a take-home assignment, with no class discussion. Another approach is to have them work on this in class for a short period of time (10-20 minutes) and then they take it home to complete it. I prefer they start this activity in class, because they can get a better feel for expectations and it helps them know whether or not they are on the right track.
- Problem 5 has students listing function characteristics. I don't like to tell them how many attributes they have to list, because I want to see how they think about functions and I don't want to limit their ideas.[1] The tradeoff though is that some students only list one attribute for each part. Typically when I grade this problem, I want to see three things listed for each part.
- Because this is a long activity requiring thoughtfulness, students can be given the option to work with a partner. I have always provided this option; however, most students choose to complete this individually.

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[1] This sort of approach came about as a result of taking part in the Creativity in Calculus project. `http://www.creativityresearchgroup.com/`

1. What does $f'(x)$ represent in terms of the graph of a function f?

2. Suppose you are given two points, $(a, f(a))$ and $(b, f(b))$, on a graph of the function f. How would you find the slope of the secant line through those points?

3. Consider the function $f(x) = x^3 + 2x^2 - x - 2$.

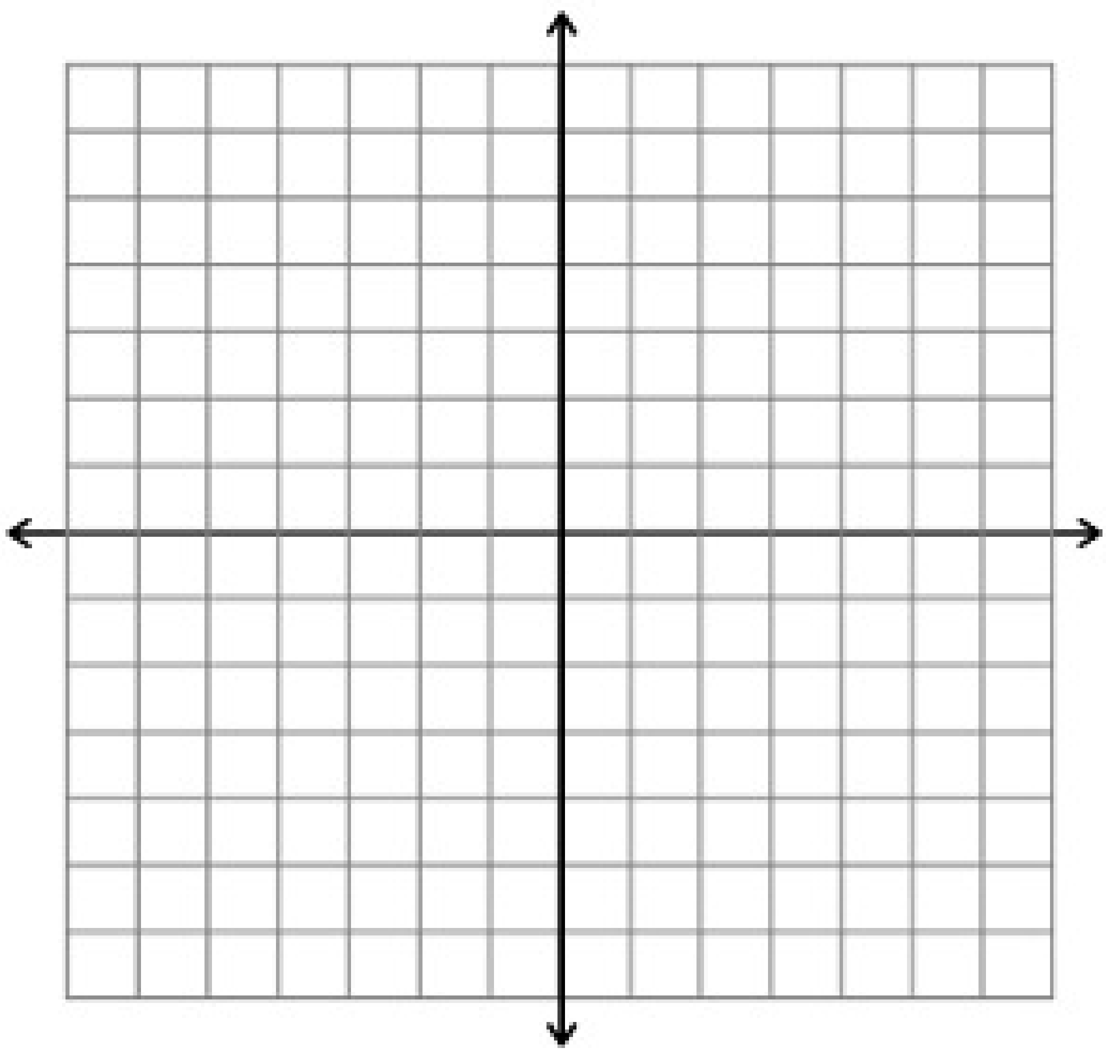
   (a) Graph f on $[-2, 1]$ on the plane given on page 2.

   (b) Describe the graph of f on $[-2, 1]$. List as many characteristics as possible.

(c) Graph the secant line through $(-2, f(-2))$ and $(1, f(1))$ on the same plane.

(d) What is the slope of the secant line?

(e) Is there a c in the interval $(-2, 1)$ such that $f'(c)$ is equal to the slope of the secant line from part (c)? Why or why not?



4. Consider the function $g(x) = \frac{1}{x^2}$.

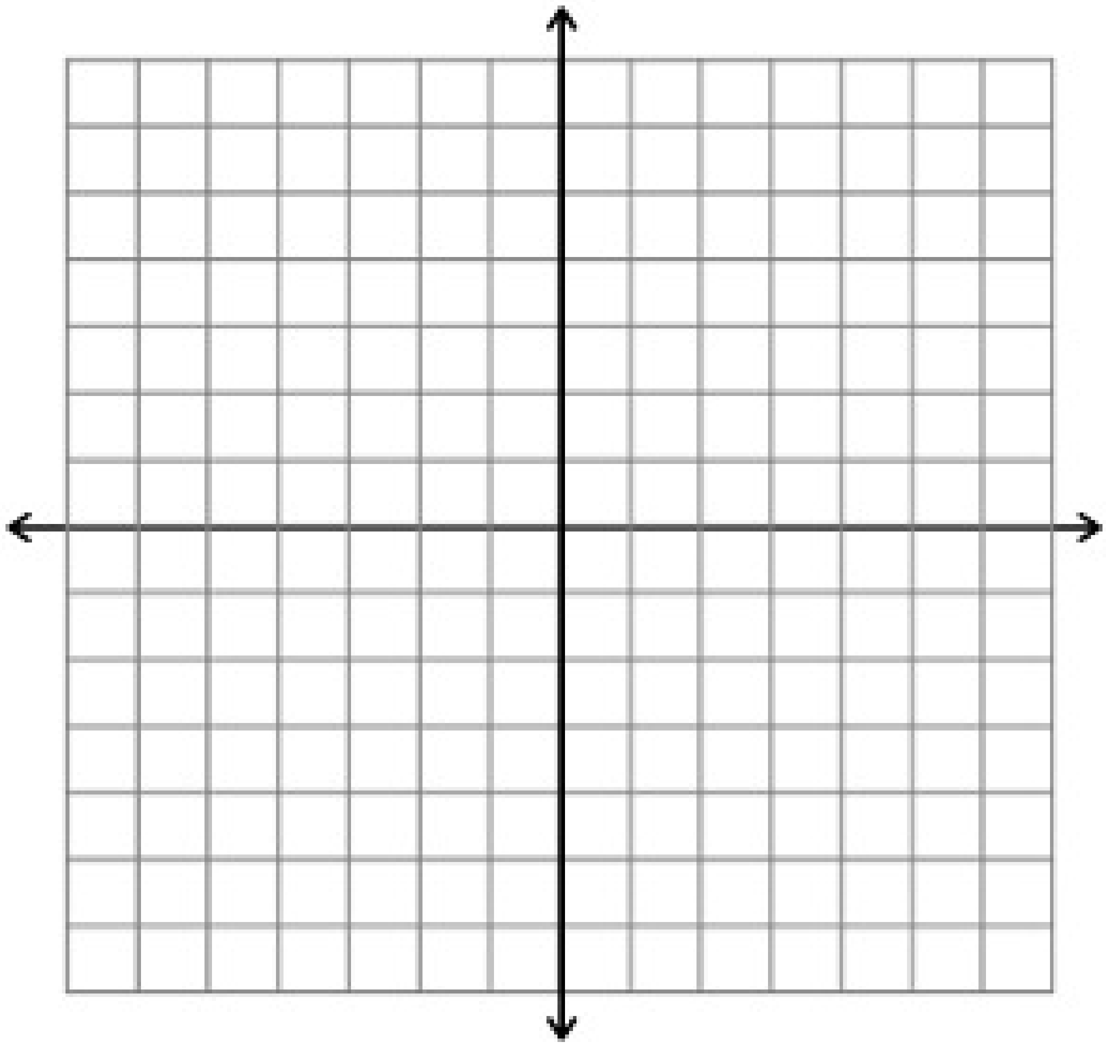
   (a) Graph g on $[-2, 2]$ on the plane given on page 4.

   (b) Describe the graph of g on $[-2, 2]$. List as many characteristics as possible.

   (c) Graph the secant line through $(-2, g(-2))$ and $(2, g(2))$ on the same plane.

   (d) What is the slope of the secant line?

   (e) Is there a c in $(-2, 2)$ such that $g'(c)$ is equal to the slope of the secant line from part (c)? Why or why not?



5. You previously listed physical characteristics of $f(x) = x^3 + 2x^2 - x - 2$ and $g(x) = \frac{1}{x^2}$.

   (a) How are the functions similar?

(b) How are the functions different?

6. Now, consider the following theorem:

Theorem 1 (Mean Value Theorem) *Suppose* $y = h(x)$ *is a continuous function on a closed interval* $[a, b]$ *and differentiable on the interval's interior* (a, b)*. Then there is at least one point* c *in* (a, b) *at which*

$$h'(c) = \frac{h(b)-h(a)}{b-a}.$$

(a) What does $h'(c)$ represent in terms of a graph?

(b) What does $\frac{h(b)-h(a)}{b-a}$ represent in terms of a graph?

(c) What does $h'(c) = \frac{h(b)-h(a)}{b-a}$ mean graphically?

(d) Based on the Mean Value Theorem, why did f (from problem 3) have a tangent line with the same slope as the secant line but g (from problem 4) did not?

(e) Graph a random function f which is continuous on the closed interval $[-3, 2]$ and differentiable on the open interval $(-3, 2)$. Graph the secant line through $(-3, f(-3))$ and $(2, f(2))$. Find an x in the interval $(-3, 2)$ such that the tangent line at x is parallel to the secant line. Graph that tangent line.

7. A consequence of the Mean Value Theorem is Rolle's Theorem.

Theorem 2 (Rolle's Theorem) *Suppose $y = h(x)$ is a continuous function on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). If $h(a) = h(b)$, then there is at least one number c in (a, b) at which*

$$h'(c) = 0.$$

(a) Why is Rolle's Theorem a consequence of the Mean Value Theorem?

(b) [2] The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Justify your answer.

[2] This question is based off of a problem found in Thomas' Calculus, 13th Edition, section 4.2, problem 13.

(c) Give an example of a function f that is continuous on a closed interval $[a, b]$ (be sure to include the interval) such that $f(a) = f(b)$ but f is not differentiable on the open interval (a, b). Using the function you created and the corresponding interval, does there exist a c in (a, b) such that $f'(c) = 0$? Does Rolle's Theorem apply? Why or why not?

(d) Give an example of a non-constant function f and domain which satisfies the assumptions of Rolle's Theorem. Then, find a value of c in your domain such that $f'(c) = 0$.