LESSON TITLE: Completing the Square

OVERVIEW: This lesson explores the topic of completing the square through group work and algebra tiles. This hands-on approach helps students visualize quadratic expressions and their transformations, helping them deepen their understanding of perfect square trinomials and how they are applied in solving quadratic equations, graphing parabolas, finding the center and radius of a circle, and deriving the Quadratic Formula. The activity encourages discovery, discussion, and practical problem-solving, aiming to enhance students' algebraic and collaborative skills.

PREREQUISITE IDEAS AND SKILLS

- Students should know that the area of a square is equal to the square of its side length.
- Students should be able to multiply (i.e., expand) algebraic expressions and simplify them by combining like terms.
- Students should be able to factor simple quadratic expressions.
- Students should know about transformations of graphs of functions and know how to graph the basic function $y = x^2$.
- Students should recognize the standard form for the equation of a circle, which is $(x-h)^2 + (y-k)^2 = r^2$, and know that this circle has the center point (h, k) and radius r.

MATERIALS NEEDED TO CARRY OUT THE LESSON

- Sets of algebra tiles for each group (available for purchase on Amazon or other retailers at around $7-$12 per set) OR you can print and cut out paper versions on either white or colored paper from this website: https://mathbits.com/MathBits/AlgebraTiles/Algebra%20Tile%20Template.pdf OR you can have students use the digital version found one this website: https://mathsbot.com/manipulatives/tiles
- Printed copies of the "Completing the Square" worksheet for each student or group
- Whiteboard or chalkboard for class discussions and demonstrations

CONCEPTS TO BE LEARNED/APPLIED

When first learning the technique of completing the square, students often encounter algebraic expressions that, while appearing different, yield identical results for any given input. For example, the expressions $f(x) = x^2 + 10x + 28$ and $g(x) = (x+5)^2 + 3$ are understood to be equivalent because they produce the same outputs for any given input (such as both evaluating to 39 when $x = 1$). This common numerical interpretation, however, often leaves the underlying reasons for their equivalence unclear, making the method seem like simply a mathematical trick. Our activity aims to bridge this gap by introducing a descriptive interpretation of equivalence, where students come to understand that these expressions are not just numerically identical but fundamentally describe the same geometric reality. Through the use of algebra tiles, this lesson helps students visualize how both $f(x)$ and $g(x)$ actually represent the same total area. This helps to ground their understanding in a tangible concept of equivalence that extends beyond simple numerical equality.

The following are descriptions of the specific concepts to be learned/applied:

- Students will learn the method of completing the square to transform quadratic expressions of the form $x^2 + bx + c$ into the form $(x+h)^2 + k$.

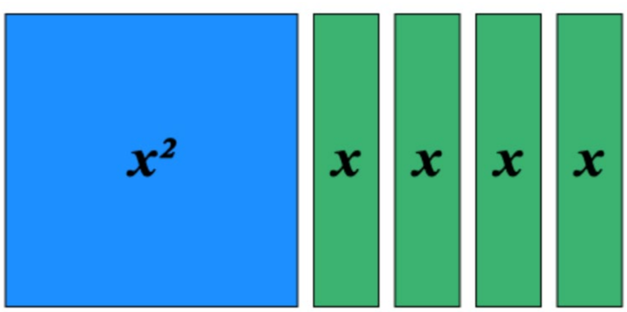
- Students will use algebra tiles to connect geometric figures to algebraic expressions, fostering a deeper understanding of perfect square trinomials and quadratic expressions, in general.
- Students will apply the method of completing the square to solve quadratic equations.
- Students will derive the Quadratic Formula through completing the square.
- Students will use completing the square to graph a parabola using transformations (i.e., without using the formula for the axis of symmetry).
- Students will use completing the square to determine the center and radius of a circle from its expanded equation.
- Students will use completing the square to understand how the Quadratic Formula can be derived.

INSTRUCTIONAL PLAN

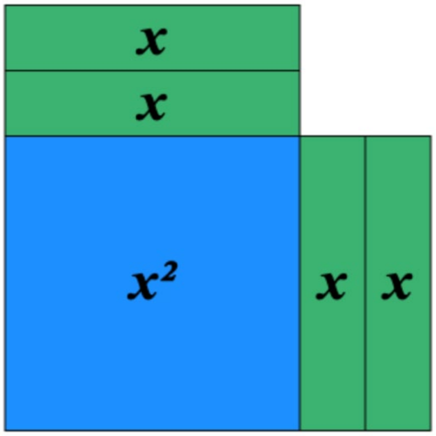
Introduction: Completing the Square

At the start of the activity, students are organized into groups of 3 or 4 with each group given a set of algebra tiles and a worksheet. The large square tiles represent x^2, the long rectangles represent $x \times 1$, and the small unit squares represent the constant term 1 . An initial demonstration by the instructor shows how to form a larger square using these tiles, beginning with one x^2 tile and a selection of x-tiles. For example, starting with an x^2 tile and 6 x-tiles, the instructor poses the question of how many small squares are needed to "complete the square." The images below were created and copied from: https://mathsbot.com/manipulatives/tiles (On the website all pieces are in a vertical orientation. The rotate button at the top of the website will put the selected piece in a horizontal orientation.)

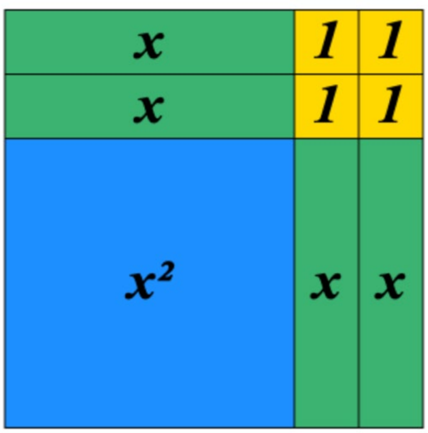
Step 1: Tiles shown when task is posed.



Step 2: Tiles shown re-arranged.



Step 3: Figure with the small unit squares added to form a larger square.



Group Work

Next, have groups do a hands-on exploration. Task groups with creating larger squares using one x^2 tile and a varying even numbers of x-tiles, noting the importance of symmetry which necessitates an even number of x-tiles. (A later discussion will be used to address the case of an odd number of x variables.) As they experiment, each group records their findings in a table, noting the relationship between the number of x-tiles and the required number of small squares to complete the square. The structured exploration (with record-keeping) is designed to encourage discussion and discovery among students, helping them recognize the patterns that lead to the algebraic technique of completing the square. Students usually first notice that half of the x-tiles go to the right and the other half go on top of the x^2 tile. Next, students, with prompting, should be able to determine that $\left(\frac{x}{2}\right)^2$ small individual unit squares are needed to complete the large square. For example, if given an x^2 tile with 10 x-tiles, students should be able to notice that 25 small squares are needed to complete the larger square, leading to a geometric representation of the algebraic expression $x^2 + 10x + 25$. The instructor should help the group recognize that the large square is also represented by $(x + 5)^2$ by considering the area of the larger square by looking at its side lengths.

This can be done by having students complete parts 1 and 2 of the worksheet in their groups.

Class Discussion

Next, the activity can transition to a classroom discussion of the observed patterns and the relationship to algebraic expressions to make sure that students are understanding.

Introduction: Perfect Square Trinomials

The concept of "perfect square trinomials" is introduced next, and $x^2 + 6x + 9$ may be used as an example. The instructor should emphasize that the word "trinomial" is used because of the three terms, and it's a "perfect square" because it is the square of a binomial (in this case, it's the square of $(x + 3)$).

The correlation between the geometric activity with the tiles and algebraic manipulation is then explored. Taking an expression such as $x^2 + 10x$, the instructor demonstrates its transformation into $(x + 5)^2 - 25$, while highlighting the algebraic steps that mirror the tile arrangements. In this example, 25 tiles must be removed from the $(x + 5)^2$ large square formation in order to arrive at the geometric formation of $x^2 + 10x$. When it comes to the algebra, the idea of "both adding and subtracting" 25 will be helpful for this example, so that $x^2 + 10x = x^2 + 10x + 25 - 25 = (x + 5)^2 - 25$.

Class Discussion

Next, common challenges to completing the square, like odd coefficients, are addressed in a discussion format. For an expression such as $x^2 + 7x$, students are asked for ideas about what needs to be done to complete the square (i.e., adding $\left(\frac{7}{2}\right)^2 = \frac{49}{4}$). There can also be a discussion of how this relates to the algebra tiles (using 3-and-one-half x-tiles on each side). Another common challenge is negative coefficients, as in the case of $x^2 - 6x$. In this case, using the algebra tiles is a bit trickier, however a similar idea can be used with a drawing on the board. For example, $x^2 - 6x$ can be drawn by first drawing a large $x \times x$ square, and then coming inward by a length of 3 on both sides, an inner square can be drawn with side lengths $x - 3$. With both of these challenges (odd coefficients and negative coefficients), it should be emphasized that to have a perfect square trinomial, the constant on the end must be the square of half the coefficient of x. In other words, to complete the square with $x^2 + kx$ the constant term $\left(\frac{k}{2}\right)^2$ must be added. This is true whether k is even or odd or positive or negative. This should all be drawn out in a classroom discussion format, rather than the instructor simply giving a lecture.

Group Work

Students return to their worksheets in their groups, where in Part 3 they are asked to solve the equation $x^2 + 10x + 14 = 0$ using what they have learned about completing the square. Instructors should emphasize that students are not allowed to use the Quadratic Formula (except to check their work at the end). The instructor should circulate among the groups and allow for some productive struggle, as giving too much help initially may be detrimental. If students are struggling for too long, the instructor can give hints, such as subtracting 14 from both sides to get $x^2 + 10x = -14$. If students are still stuck after a bit more struggle, the instructor can give the hint to complete the square on the left-hand side. However, it would be more preferable if students come up with this idea on their own. Finally, students should remember that when solving $(x + 5)^2 = 11$ by taking a square root, the "plus or minus" must be used. Students then work the next example, $x^2 - 6x + 5 = 0$, and check their answer by completing the square.

Students then complete Part 4, where they apply completing the square to two scenarios: (1) Graphing a parabola ($y = x^2 - 6x + 8$) by completing the square and using transformations of the graph of $y = x^2$ (i.e., without using the axis of symmetry), and (2) Finding the center and radius of a circle given in expanded form ($x^2 + 4x + y^2 + 6y - 12 = 0$) by completing the square with both x and y.

Class Discussion

In Part 5 of the worksheet, students are asked if they know where the Quadratic Formula comes from.

The instructor should lead a whole class discussion ensues, asking students if they have ideas for how to solve a general equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Ideally, an ambitious student would have some ideas, and the instructor may ask if the student is willing to try solving it on the board (with help from classmates and the instructor). If no students are willing to attempt it, the instructor can walk the class through the steps to solving it (i.e., to deriving the Quadratic Formula) using completing the square. Click here for a video which goes through the method of completing the square and culminates (in the last 2 minutes and 30 seconds) with a derivation of the Quadratic Formula. A link to this video may be given to students if the instructor chooses.

MIP COMPONENTS OF INQUIRY

This section outlines how our activity will meet the Mathematical Inquiry Project (MIP) criteria for active learning, meaningful applications, and academic success skills.

Active Learning:

This activity engages students in active learning by guiding them through the process of completing the square in a hands-on, exploratory manner. Rather than being presented with the method as a pre-packaged formula, students engage in activities that require them to manipulate algebra tiles, drawing parallels between physical tile arrangements and algebraic structures. For instance, when tasked with forming a larger square using a combination of x^2, x, and constant tiles, students must determine the necessary additions to complete the square, mirroring the algebraic process of adding and then subtracting a specific value to create a perfect square trinomial. This tactile experience, coupled with group discussions, prompts students to actively select, perform, and evaluate actions that reflect the structures of the underlying concepts to be learned, thus strengthening their understanding and retention.

Meaningful Applications:

This activity uses meaningful applications by connecting the technique of completing the square to familiar mathematical contexts, thus reinforcing its utility across various areas of mathematics. For example, students apply the method to graphing parabolas, a concept they are likely familiar with, by finding the vertex form of quadratic functions and using transformations of graphs, which is another concept they have previously worked with. Students also use completing the square to rewrite the equations of circles, allowing them to easily identify the center and radius from the standard form of the equation of a circle. This ties the algebraic technique to geometric concepts they have encountered. Students also see in the activity how the method can be used to solve quadratic equations, especially when equations are not easily factorable. Finally, the activity leads students through the derivation of the quadratic formula using completing the square. This not only demystifies a formula that students often memorize without understanding where it comes from, but it also highlights the foundational role of the technique of completing the square in algebra. By linking completing the square to these various contexts, students recognize how broadly the method can be applied to other areas of math that they are familiar with.

Academic Success Skills:

This activity will allow students to make mathematical discoveries themselves as opposed to simply being given step-by-step methods and formulas. Moreover, students will see the usefulness of their productive struggle as they see how their newfound technique can be applied in various contexts. This should help empower them to see themselves as more capable of developing mathematics and not just someone who must just memorize the equations and rules given from an authority figure.