The derivative at a point, Part 1 – Instructor Notes

Preparation: A bolt is fired from a crossbow straight up into the air with an initial velocity of 49 m/s. Accounting for wind resistance proportional to the speed of the bolt, its height above the ground is given by the equation $h(t) = 7350 - 245t - 7350e^{-t/25}$ meters (with t measured in seconds). Throughout this activity, you will approximate the speed when $t = 2$ seconds.

A. Draw a *full-page* picture of the physical context in three different configurations of the crossbow bolt *overlayed*. This will help illustrate how the two quantities are changing near the time of interest $t = 2$ seconds. You will redraw and add to this picture during class.

   See answer to Question #1 below.

B. What happens to the changes in the height as the time increases by constant amounts? Is the rate of change (speed) constant, increasing or decreasing?

   Answer. The rate that we are approximating is the instantaneous speed of the bolt two seconds after it was fired. Specifically, this is the rate of change of the height of the bolt with respect to the time elapsed since it was fired. We are measuring the height, h, of the bolt in meters and the time, t, in seconds. The speed of the bolt is a variable quantity, but we cannot compute it directly from the formula relating h and t provided. Determining how the instantaneous speed can be inferred from this relationship is the point of this assignment. Plugging numbers into h or looking at a graph, we clearly see that the bolt is still moving up 2 seconds after it has been fired. Thus the bolt is slowing down (due to both gravity and air resistance). While the bolt is moving up, h increases as t increases, however h changes by less and less for fixed increments of time. Once the bolt reaches its highest point, the height h starts to decrease as time passes. It will speed up, so h will decrease by greater amounts for fixed increments of time.

C. Draw a *full-page* graph showing the relationship between the two quantities involved in the instantaneous rate that you are asked to approximate. Add a point for each of the configurations you drew in your picture from A. Represent the changes in both quantities on your graphs as the length of short line segments. You will redraw and add to this graph during class.

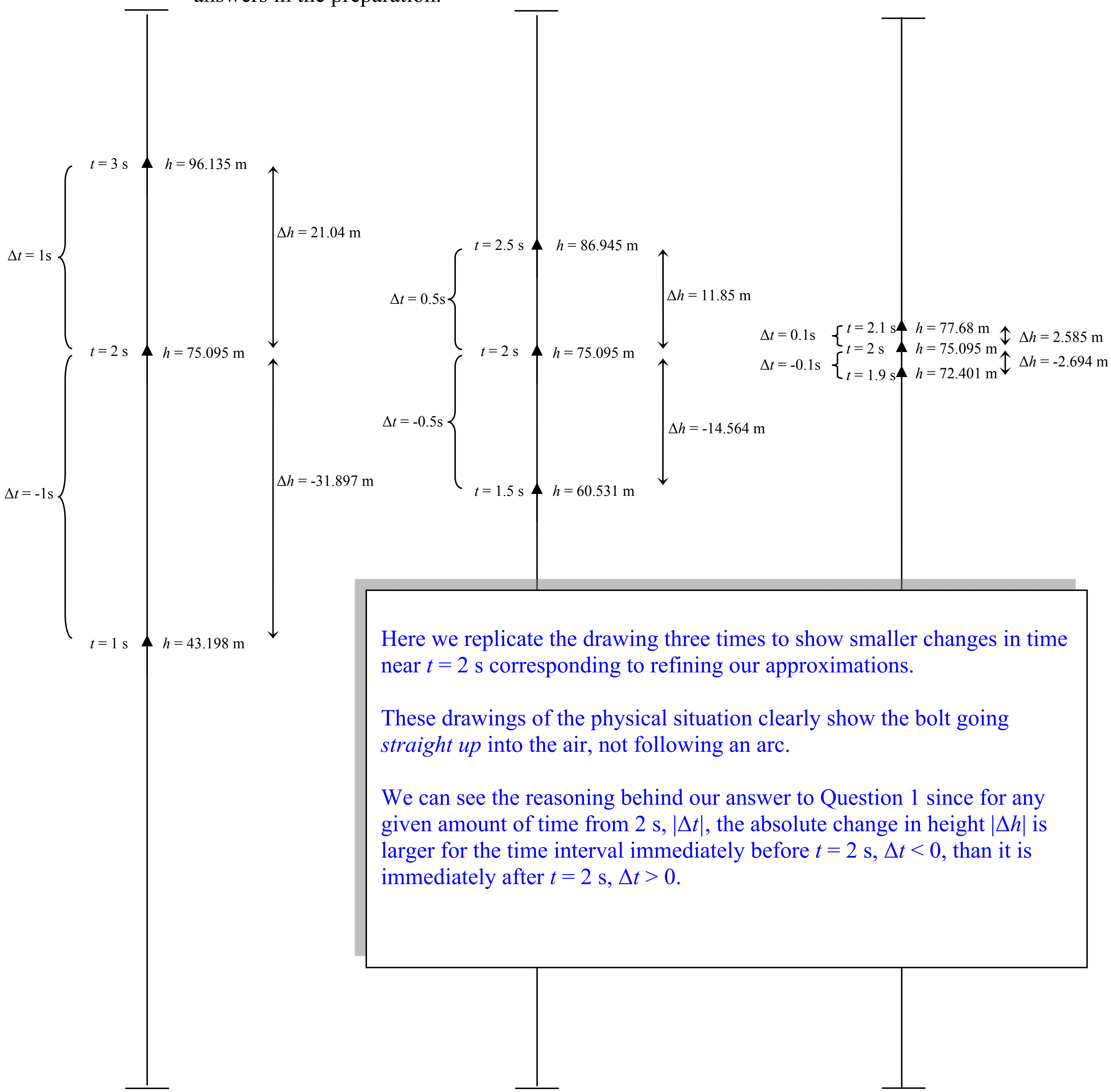
   See answer to Question #2 below.

D. On your picture *and* your graph illustrate and label the changes in the relevant quantities to support your answer to B (using both Δ-notation and numerical values).

   See answer to Questions #1 and #2 below.

The derivative at a point, Part 1

1. Draw a *full-page* picture of several "snapshots" showing
   a. The bolt at time $t = 2$ seconds.
   b. The bolt at other times from your preparation work.
   c. Changes in height and time (using both Δ-notation and numerical values) to support your answers in the preparation.



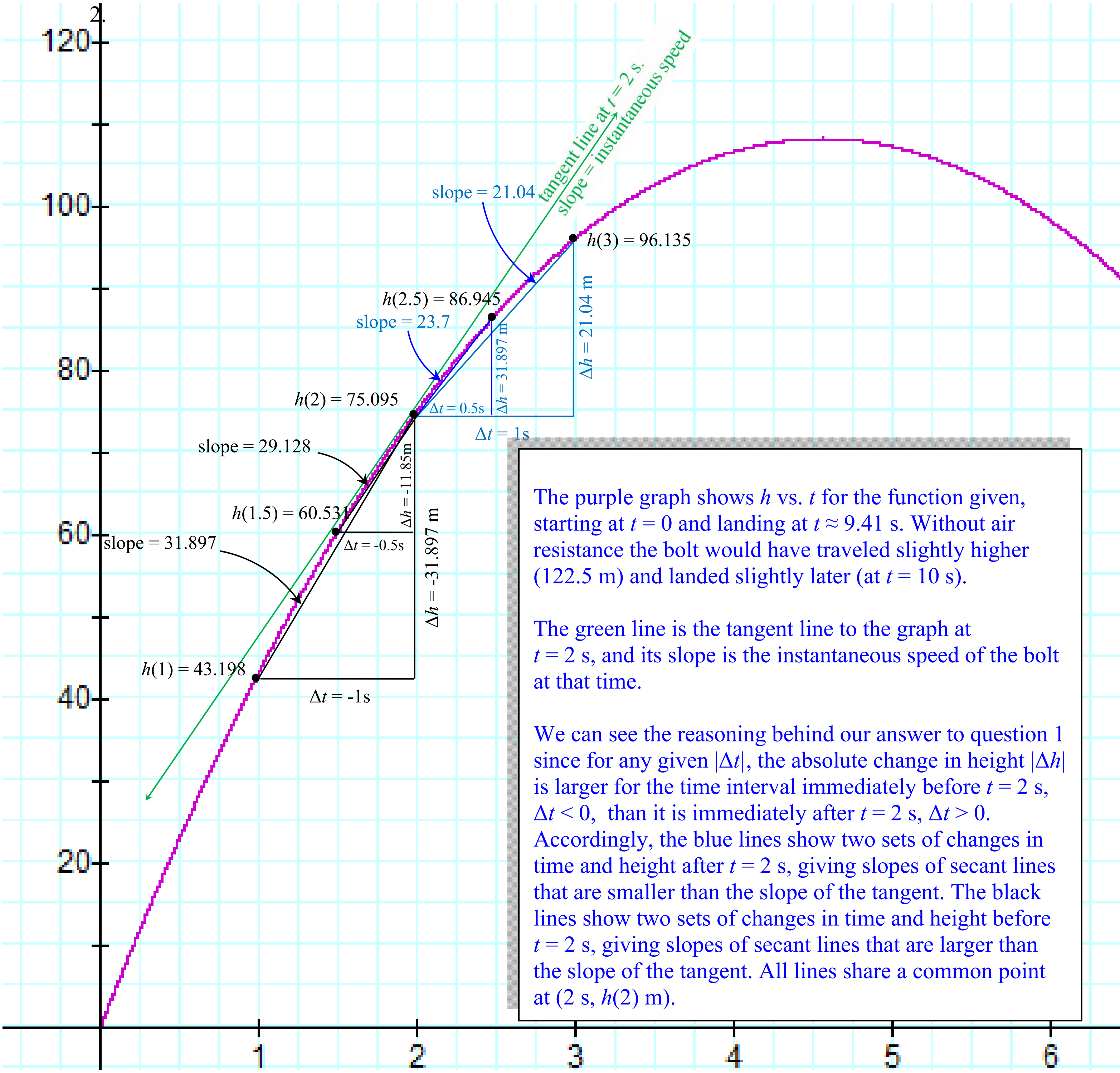
Here we replicate the drawing three times to show smaller changes in time near $t = 2$ s corresponding to refining our approximations.

These drawings of the physical situation clearly show the bolt going *straight up* into the air, not following an arc.

We can see the reasoning behind our answer to Question 1 since for any given amount of time from 2 s, $|\Delta t|$, the absolute change in height $|\Delta h|$ is larger for the time interval immediately before $t = 2$ s, $\Delta t < 0$, than it is immediately after $t = 2$ s, $\Delta t > 0$.

2. Draw a *full-page* graph showing
   a. Several points corresponding to the bolt at time $t = 2$ seconds and three or four nearby times.
   b. Algebraic and numerical representations of the height and time at the points in Part a
   c. Changes in height and changes in time starting from time $t = 2$ seconds (using both Δ-notation and numerical values)

2.



3. In this question, you will provide details about *what you have been asked to approximate*.
   a. Describe what you have been asked to approximate using language about the crossbow bolt.
   b. Define a variable to represent this unknown value algebraically. What units will be attached to it?
   c. Represent this unknown value that you are approximating on your graph. Label it with your chosen variable. What attribute of the object that you added to your graph corresponds to the unknown value you are approximating?

Answer. We have been asked to approximate the instantaneous speed of the bolt 2 seconds after it was fired. We will call this speed $v(2)$ meters per second, and it is represented by the slope of the tangent line in the graph.

4. In this question, you will provide details about *approximations* to your instantaneous speed.
   a. Compute 3 average rates of change that approximate the speed of the crossbow bolt at time $t = 2$ seconds.
   b. Using language about the crossbow bolt, explain the physical meaning of one of your average rates of change.
   c. Write an algebraic expression showing someone how to compute these average rates in general.
   d. Represent the 3 approximations on your graph. Label them with numerical values. What attribute of the things that you added to your graph corresponds to the approximation values?

Answer. We can approximate the instantaneous speed with the average speed of the bolt during short periods of time near $t = 2$ s.

The average speed during the one-second intervals immediately before and after $t = 2$ s are

$$\frac{\Delta h}{\Delta t} = \frac{43.198\text{ m} - 75.095\text{ m}}{1\text{ s} - 2\text{ s}} = \frac{-31.897\text{ m}}{-1\text{ s}} = 31.897\ \text{m/s}$$

and

$$\frac{\Delta h}{\Delta t} = \frac{96.135\text{ m} - 75.095\text{ m}}{3\text{ s} - 2\text{ s}} = \frac{21.04\text{ m}}{1\text{ s}} = 21.04\ \text{m/s}$$

The average speed during the half-second intervals immediately before and after $t = 2$ s are

$$\frac{\Delta h}{\Delta t} = \frac{60.531 \text{ m} - 75.095 \text{ m}}{1.5 \text{ s} - 2 \text{ s}} = \frac{-14.564 \text{ m}}{-0.5 \text{ s}} = 29.128 \text{ m/s}$$

and

$$\frac{\Delta h}{\Delta t} = \frac{86.945 \text{ m} - 75.095 \text{ m}}{2.5 \text{ s} - 2 \text{ s}} = \frac{11.85 \text{ m}}{0.5 \text{ s}} = 23.7 \text{ m/s}$$

The average speed of 23.7 m/s for the bolt (which is slowing down as it goes up) *means* that another object would have to travel at a *constant speed* of 23.7 m/s in order to cover the *same distance*, 11.85 m, *in the same time*, 0.5 s.

Most generally we can express the approximations as

$$\frac{h(2+\Delta t) - h(2)}{\Delta t} \text{ for various choices of } \Delta t \text{ (positive or negative).}$$

Graphically, these average speeds correspond to the slopes of secant lines (since Δh corresponds to the "rise" and Δt corresponds to the "run" in the graph).

5. In this question, you will identify both *underestimates* and *overestimates* for the requested instantaneous rate.

   a. If you have not already found both underestimates and overestimates and represented them on your graph, do so.

   b. Using only language about your physical context (not your graph), explain how you know these are in fact underestimates and overestimates.

   c. Explain how your explanation from Part b can be seen on both the picture of the situation and on the graph.

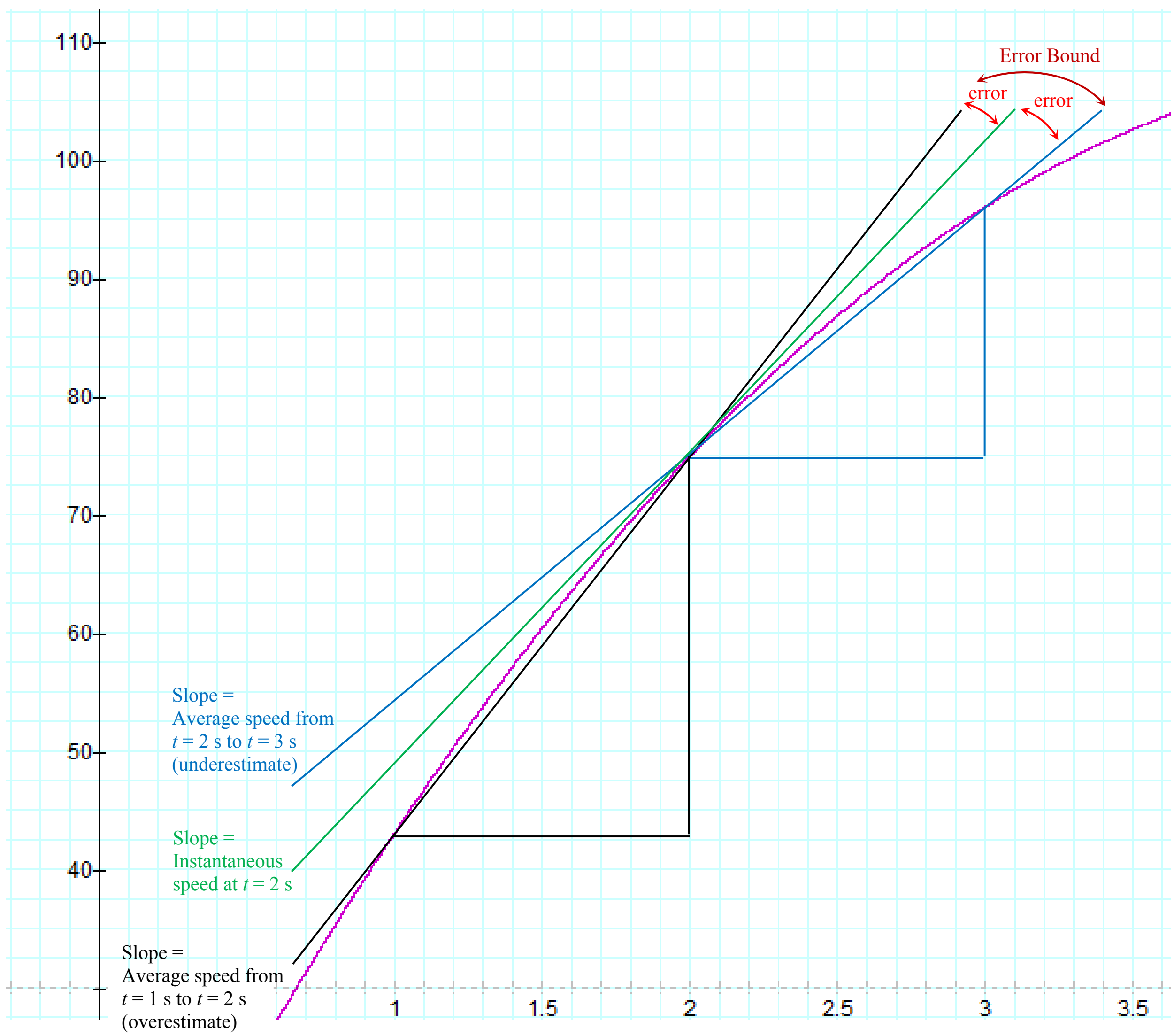
Answer. Since the bolt is slowing down, it is moving faster during the entire half-second prior to $t = 2$ s than it is at $t = 2$ s. Thus the average speed of 29.128 m/s is an overestimate for $v(2)$. The bolt is moving slower during the entire half-second after $t = 2$ than it is at $t = 2$ s. Thus the average speed of 23.7 m/s is an underestimate for $v(2)$. (Note that it is not enough to justify whether something is an underestimate or overestimate by simply noting that one approximation is smaller than the other – without additional information we could have two underestimates or two overestimates.) We see that the average speeds prior to $t = 2$ s will give overestimates in the picture of the situation because the changes in height are decreasing. We see that the average speeds after to $t = 2$ s will give underestimates in the graph because the slopes are decreasing.

6. In this question, you will identify and represent the *errors* in your approximations.
   a. Give an algebraic representation of the errors for both an underestimate and an overestimate.
   b. Explain how these errors are represented graphically. Add and label the errors on your graph.

Answer. Since we have algebraic and graphical representations for the unknown instantaneous speed at $t = 2$, we can also represent the error algebraically and graphically. Specifically, algebraically, the errors for the two approximations computed above are

$$\left|v(2)-29.128\right| \text{ and } \left|v(2)-23.7\right| \text{ or more generally, } \left|v(2)-\frac{h(2+\Delta t)-h(2)}{\Delta t}\right|$$

Graphically, the errors are the differences between the slope of the tangent line and the slopes of the secant lines as shown in the graph below (here we use a relatively large $|\Delta t| = 1$ s to exaggerate the difference in slopes so that they are visible).



7. In this question, you will identify and represent the *error bounds* in your approximations.
   a. Find an error bound for one of your approximations. Justify your answer.
   b. Explain how this error bound is represented graphically. Add and label the error bound on your graph.
   c. What is the resulting range of possible values for your instantaneous rate? Explain how this range is represented graphically.

Answer. For $|\Delta t| = 0.1$ s, we get an underestimate of 25.854 m/s and an overestimate of 26.940 m/s, so the error bound is 1.086 m/s. Algebraically we would write $|v(2)-25.854|<1.086$ and $|v(2)-26.940|<1.086$. Graphically the error bound is the difference between the slopes of the two secant lines, one slope being an overestimate and the other being an underestimate. The range of possible values for the instantaneous speed is $25.854<v(2)<26.940$. This range of values is the range of possible slopes for the (green) tangent line to the graph at $t = 2$.

8. Find an approximation accurate to within the error bound given in your problem. Show and explain all of your work.

Answer. For $|\Delta t| = 0.01$ s, we get an underestimate of 26.342 m/s and an overestimate of 26.451 m/s, so the range of possible values for the instantaneous speed is $26.342<v(2)<26.451$, and the error bound is 0.11 m/s. Thus our approximations are not quite accurate enough.

For $|\Delta t| = 0.005$ s, we get an underestimate of 26.369 m/s and an overestimate of 26.423 m/s, so the range of possible values for the instantaneous speed is $26.369<v(2)<26.423$, and the error bound is 0.054 m/s, so this is within the desired 0.1 m/s error bound.

Note: Given any desired error bound we could always find a sufficiently accurate approximation by repeating the process of using smaller and smaller values of Δt to compute the average speeds $\dfrac{h(2+\Delta t)-h(2)}{\Delta t}$ before and after $t = 2$ seconds. Find the difference between the two to compute the error bound, $\left|v(2)-\dfrac{h(2+\Delta t)-h(2)}{\Delta t}\right|<\text{EB}$, and see if that is small enough. If so, then either the underestimate or the overestimate will work. If not, pick a smaller Δt and repeat the process. Eventually the error bound will get as small as desired.