1. What does $f'(x)$ represent in terms of the graph of a function $f$?

$f'(x)$ represents the slope of the tangent line to $f$ @ $x$

$f'(x)$ represents the slope of the curve $f$ @ $x$.

2. Suppose you are given two points, $(a, f(a))$ and $(b, f(b))$, on a graph of the function $f$. How would you find the slope of the secant line through those points?

$$\frac{f(b)-f(a)}{b-a} = \frac{f(a)-f(b)}{a-b}$$

only one is necessary (don't need to include both)

3. Consider the function $f(x) = x^3 + 2x^2 - x - 2$.

(a) Graph $f$ on $[-2, 1]$ on the plane given on page 2.

(b) Describe the graph of $f$ on $[-2, 1]$. List as many characteristics as possible.

$f$ is continuous on the closed interval $[-2,1]$

$f$ has an absolute maximum (@ $x = \frac{-2-\sqrt{7}}{3}$)

$f$ has an absolute minimum (@ $x = \frac{-2+\sqrt{7}}{3}$)

$f$ is concave down on $(-2, -\frac{2}{3})$

$f$ is concave up on $(-\frac{2}{3}, 1)$

$f$ is increasing on $[-2, \frac{-2-\sqrt{7}}{3}] \cup [\frac{-2+\sqrt{7}}{3}, 1]$

$f$ is decreasing on $[\frac{-2-\sqrt{7}}{3}, \frac{-2+\sqrt{7}}{3}]$

scratch work:

$f'(x) = 3x^2 + 4x - 1$

$f'(x) = 0$

$3x^2 + 4x - 1 = 0$

$x = \frac{-4 \pm \sqrt{16 - 4(3)(-1)}}{6}$

$= \frac{-4 \pm \sqrt{28}}{6}$

$= \frac{-4 \pm 2\sqrt{7}}{6} = \frac{-2 \pm \sqrt{7}}{3}$

~~$f'(x)$ undefined~~

cp cp

$-2$ $\frac{-2-\sqrt{7}}{3}$ $\frac{-2+\sqrt{7}}{3}$ $1$

local max local min

$f''(x) = 6x + 4$

$f''(x) = 0$

$6x + 4 = 0$

$x = -\frac{2}{3}$

~~$f''(x)$ undefined~~

$-2$ $-\frac{2}{3}$ $1$

(c) Graph the secant line through $(-2, f(-2))$ and $(1, f(1))$ on the same plane.

(d) What is the slope of the secant line?

$$\frac{f(-2)-f(1)}{-2-1} = \frac{0-0}{-3} = \boxed{0}$$

(e) Is there a $c$ in the interval $(-2, 1)$ such that $f'(c)$ is equal to the slope of the secant line from part (c)? Why or why not?

Yes, there are two places where $f'(c) = 0$.

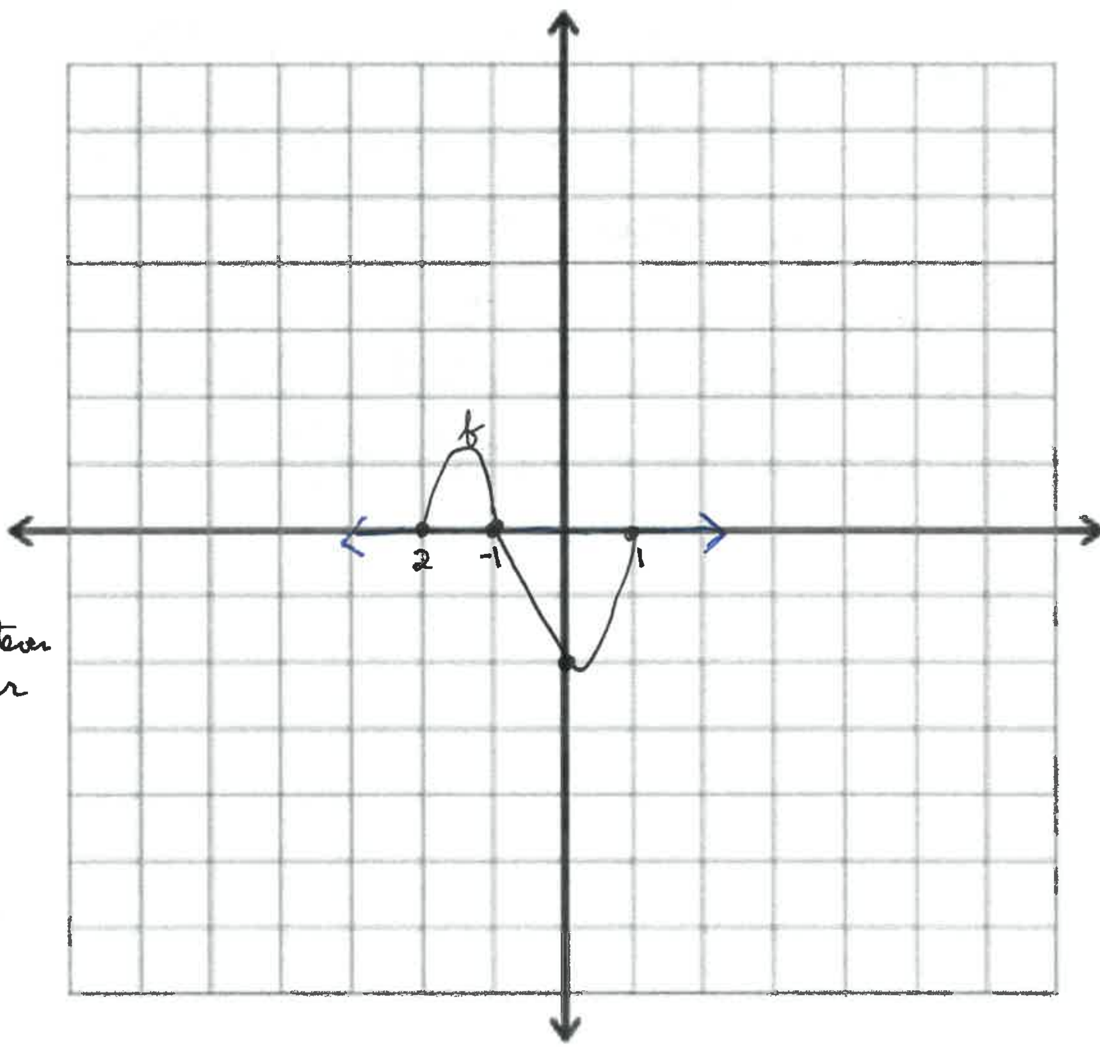
not quite all there answer

On 3b, within the scratch-work section, $f'(x)$ was computed and the x-values were found such that $f'(x)=0$.

ideal answer

Since $f$ is a polynomial, it is smooth, so it won't have any corners or cusps. We do see that $f$ switches from increasing to decreasing and decreasing to increasing w/in the domain. Thus, where the switch occurs, $f'(x)=0$.

okay to treat grid 1x1's, $\frac{1}{2} \times \frac{1}{2}$'s, or whatever size they prefer




secant line is the x-axis

4. Consider the function $g(x) = \frac{1}{x^2}$.

(a) Graph $g$ on $[-2, 2]$ on the plane given on page 4.

(b) Describe the graph of $g$ on $[-2, 2]$. List as many characteristics as possible.

g is discontinuous on $[-2,2]$
$g(0)$ is undefined
g is increasing on $[-2,0)$
g is decreasing on $(0,2]$
g is concave up on $(-2,0) \cup (0,2)$
g has an absolute minimum @ $y=\frac{1}{4}$ $(x=\pm 2)$
g has no local extrema

(c) Graph the secant line through $(-2, g(-2))$ and $(2, g(2))$ on the same plane.

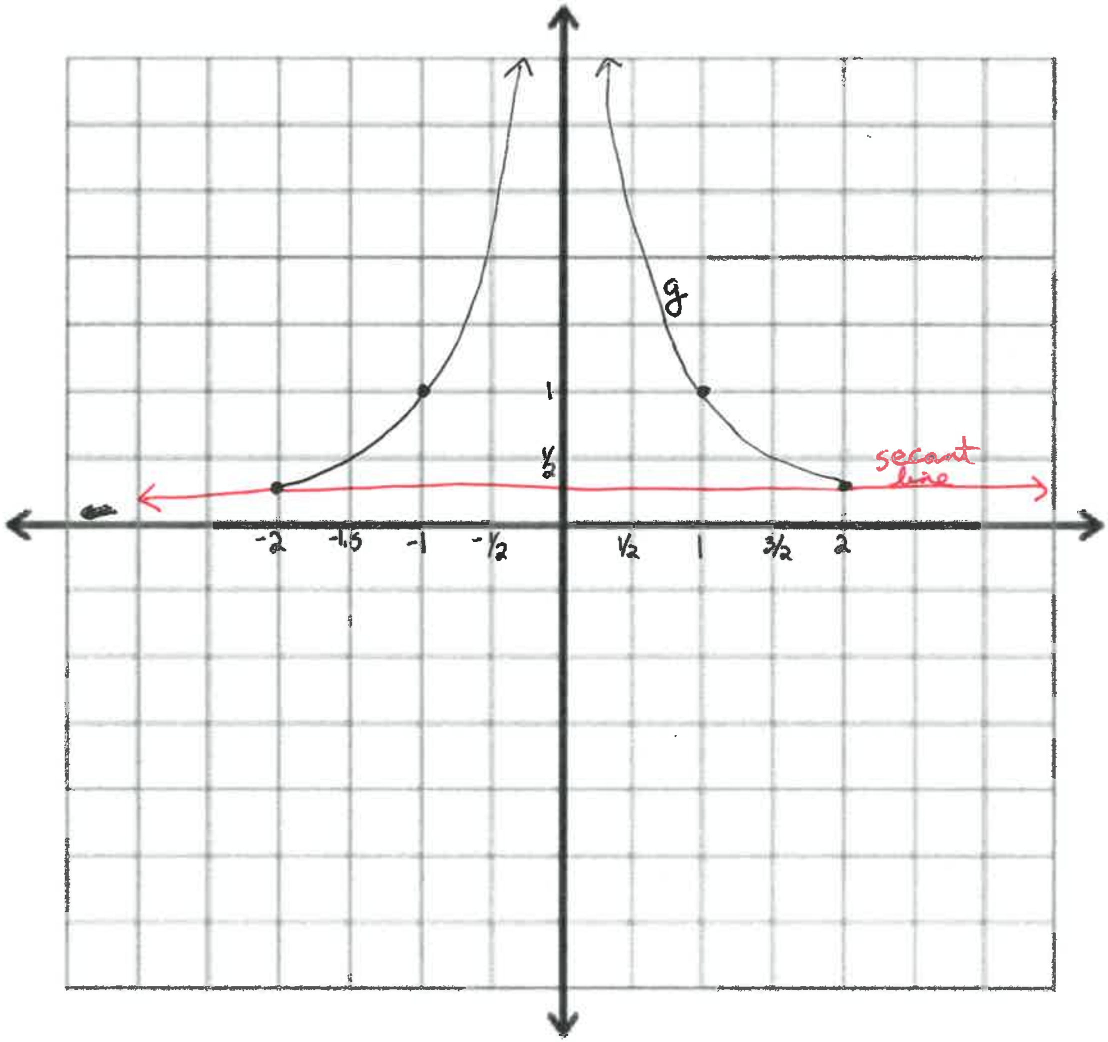
(d) What is the slope of the secant line?

$$\frac{g(2)-g(-2)}{2-(-2)} = \frac{\frac{1}{4}-\frac{1}{4}}{4} = \boxed{0}$$

(e) Is there a $c$ in $(-2, 2)$ such that $g'(c)$ is equal to the slope of the secant line from part (c)? Why or why not?

No, while g does switch from increasing to decreasing around $x=0$, $g(0)$ is undefined. Thus, $g'(0) \neq 0$.

Concavity also does not change, so we don't have to consider inflection points that also correspond to $g'(x)=0$.




5. You previously listed physical characteristics of $f(x) = x^3 + 2x^2 - x - 2$ and $g(x) = \frac{1}{x^2}$.

(a) How are the functions similar?

They both have absolute minimums.
They both have increasing and decreasing intervals.
They both have concave up intervals.

(b) How are the functions different?

f has an absolute max while g does not. (g cannot have an absolute max because $\lim_{x \to 0} g(x) = \infty$.)

f has a horizontal tangent line, while g does not.

f has an inflection point, while g does not

g has one increasing interval, while f has 2.

6. Now, consider the following theorem:

**Theorem 1 (Mean Value Theorem)** *Suppose $y = h(x)$ is a continuous function on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. Then there is at least one point $c$ in $(a, b)$ at which*

$$h'(c) = \frac{h(b)-h(a)}{b-a}.$$

(a) What does $h'(c)$ represent in terms of a graph?

The slope of the curve h @ x = c.

The slope of the tangent line to h @ x = c.

(b) What does $\frac{h(b)-h(a)}{b-a}$ represent in terms of a graph?

The slope of the secant line through the points (a, h(a)) and (b, h(b))

(c) What does $h'(c) = \frac{h(b)-h(a)}{b-a}$ mean graphically?

The slope of the secant line = the slope of a tangent line to h @ x = c

The secant line is parallel to a tangent line or equal to a tangent line.

(d) Based on the Mean Value Theorem, why did $f$ (from problem 3) have a tangent line with the same slope as the secant line but $g$ (from problem 4) did not?

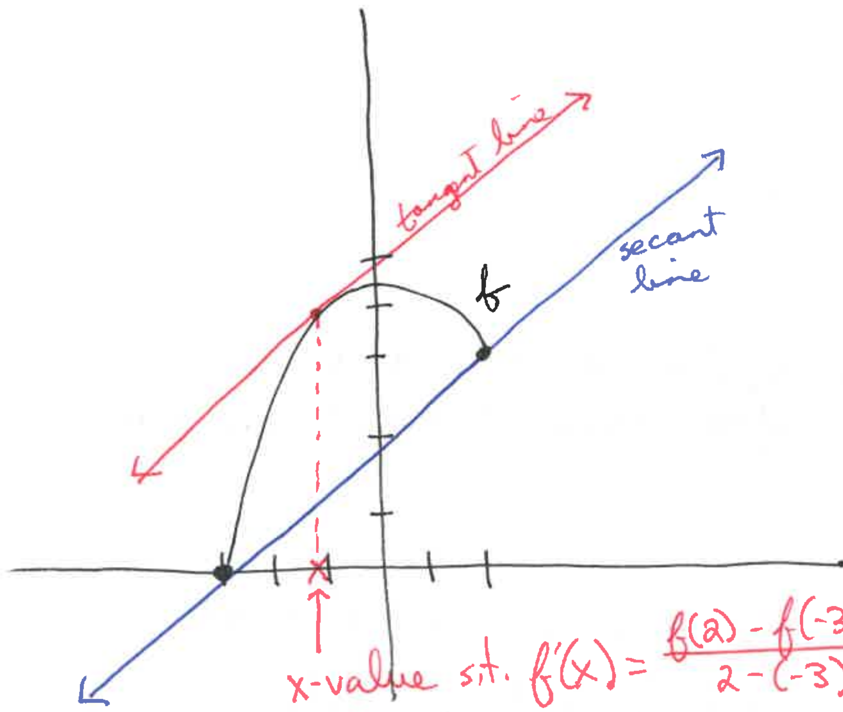
- $f$ was continuous on a closed interval and $f$ was differentiable on the open interval.
- $g$, on the other hand, was neither cont. nor differentiable. In particular, $g(0)$ and $g'(0)$ were undefined.

(e) Graph a random function $f$ which is continuous on the closed interval $[-3, 2]$ and differentiable on the open interval $(-3, 2)$. Graph the secant line through $(-3, f(-3))$ and $(2, f(2))$. Find an $x$ in the interval $(-3, 2)$ such that the tangent line at $x$ is parallel to the secant line. Graph that tangent line.

numerous correct answers

⇝ $f$ must be cont. on $[-3,2]$

⇝ $f$ must be differentiable on $(-3,2)$




x-value s.t. $f'(x) = \dfrac{f(2) - f(-3)}{2-(-3)}$

7. A consequence of the Mean Value Theorem is Rolle's Theorem.

**Theorem 2 (Rolle's Theorem)** *Suppose $y = h(x)$ is a continuous function on a closed interval $[a, b]$ and differentiable on the interval's interior $(a, b)$. If $h(a) = h(b)$, then there is at least one number $c$ in $(a, b)$ at which*

$$h'(c) = 0.$$

(a) Why is Rolle's Theorem a consequence of the Mean Value Theorem?

Rolle's Theorem has identical assumptions as the MVT, but w/ one added caviot. The mean value theorem does not specify the endpoints' y-values, but Rolle's does. So, the "if" component of Rolle's Thm still means we can apply the formula from the MVT. More specifically, we know that there exists at least 1 c in $(a,b)$ s.t. $h'(c) = \frac{h(b)-h(a)}{b-a}$. Since we know $h(b)=h(a)$, $h(b)-h(a)=0$; hence, $h'(c) = \frac{h(b)-h(a)}{b-a}$

$$= \frac{0}{b-a} = 0.$$

(b) The function

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \leq 1 \end{cases}$$

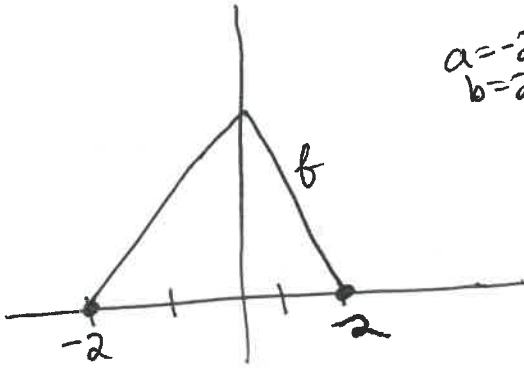
is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Justify your answer.

f is not continuous on $[0,1]$. Thus, Rolle's Theorem does not apply. This means we are not guaranteed an x-value in $(0,1)$ s.t. $f'(x)=0$.

(c) Give an example of a function $f$ that is continuous on a closed interval $[a, b]$ (be sure to include the interval) such that $f(a) = f(b)$ but $f$ is not differentiable on the open interval $(a, b)$. Using the function you created and the corresponding interval, does there exist a $c$ in $(a, b)$ such that $f'(c) = 0$? Does Rolle's Theorem apply? Why or why not?

There are numerous correct answers.

<u>1 example</u>:




$a = -2$
$b = 2$

$f(-2) = f(2) = 0$
$f'(0)$ is undefined, thus, $f$ is not differentiable on $(-2, 2)$. Hence, Rolle's Thm does not apply. Nor does there exist a $c$ in $(-2, 2)$ such that $f'(c) = 0$.

<u>another example</u>:


$a = -4$
$b = 4$

$f$ is continuous on $[-4, 4]$, but $f$ is not differentiable on $(-4, 4)$ because $f$ has a cusp. However, $f(-4) = f(4) = 0$. B/c $f$ has a cusp, Rolle's Thm does not apply. But, there still exists 2 points in $(-4, 4)$ s.t. the slope of their tangent lines equal 0.

~> this is a good problem for discussion

~> students typically provide something like the 1st example

~> students can use a graph or provide an explicit function. For example 1, students could have written $f(x) = 2 - |x|$ w/ domain $[-2, 2]$.

(d) Give an example of a non-constant function $f$ and domain which satisfies the assumptions of Rolle's Theorem. Then, find a value of $c$ in your domain such that $f'(c) = 0$.

There are multiple correct answers.

$f(x) = 4 - x^2$ on $[-1, 1]$

Assumptions satisfied

- $f$ is cont. on $[-1, 1]$
- $f$ is differentiable on $(-1, 1)$. In particular, $f'(x) = -2x$.
- $f(-1) = 4 - (-1)^2 = 4 - 1 = 3 = 4 - 1 = 4 - (1)^2 = f(1)$

$f'(c) = -2c = 0$

$\boxed{c = 0}$

→ students could have also graphed a function