The limit of a function at a point – Instructor Notes

There are two primary goals for this activity:

1. Develop a graphical, numerical and algebraic sense of the meaning of the limit of a function at a point, and
2. Develop the terminology and structure of the approximation framework that will serve as the overarching structure for everything defined in terms of a limit in the course.

Students' language about limits and approximations will initially consist of many idiosyncratic variations. So it is important to explicitly talk about and get students to agree on and verbalize the precise meanings in the approximation framework while setting others aside.

The graph of $f(x)=\dfrac{\sqrt[3]{x+3}-2}{x-5}$ has a hole. Your task is to determine the location of this hole using approximation techniques (no fancy limit computations allowed).

$$\lim_{x\to 5} f(x)=\lim_{x\to 5}\frac{\sqrt[3]{x+3}-2}{x-5}=\frac{1}{12}=0.08\overline{3}$$

You can use any other fairly simple algebraic expression where the algebraic reason for the singularity can be spotted, but the students will not likely know/remember an algebraic technique for finding the limit. The following notes are written presuming the function f given above.

Students should enter the function into their calculator and observe that there is a pixel missing from the graph. (If students are using a TI-89 or higher, they may need to set xres=1 in the window menu in order to see the hole.) If they don't have it in their window, they may need help i) noticing that there is a problem algebraically at $x = 5$ that should be included in their plot window, then ii) picking a scale where the graph fills up a good portion of the window to get a sense of what is happening. Students may think that 0/0 = 0 or 1, but looking at the graph should correct that.

Preparation Instructions: Answer the following questions *individually* and bring your write-up to class.

Expect many students to be imprecise in their lab preparation solutions. They will need to become aware of this imprecision and modify their language to speak precisely and meaningfully about what they are approximating and the approximations before they will be able to effectively move forward with the lab.

a. Draw a graph of f using an entire sheet of paper. Your graph should be drawn at a scale that gives a good sense of the *x*,*y*-coordinates of the hole. The *x* and *y* scales should be chosen so that your graph nearly extends between two diagonally opposites corners of the page.

   This is an opportunity to give students feedback about their choice of a scale when graphing a function for a specific purpose – the most important information for that purpose should be extremely clear. In this case, the graph should center roughly on the hole and be a fairly small scale around $x = 5$. The corresponding *y*-scale should be chosen so that the graph of the function uses the entire available vertical space. Calculators frequently have a zoom-fit command that will automatically restrict the values of y so that the function uses the entire vertical space for the given restriction on *x*-values for the viewing window. Tickmarks on the *y*-axis should be labeled to give a reasonable sense of the value of the *y*-coordinate of the hole.

b. Identify what **unknown numerical value** you will need to approximate. Give it an appropriate shorthand name (that is, a variable name).

   Once they spot the hole on the graph, they should notice that it is at $x = 5$, corresponding to the singularity in the algebraic expression for the function. They should recognize that the only thing they don't know about the position of the hole is its *y*-coordinate. Students may not understand what the question is asking, in which case they may need help seeing that i) finding the *y*-coordinate means finding a *number* (corresponding to the height above the *x*-axis) and ii) we often give unknown numerical values a variable name. Make sure students can articulate this meaningfully, without pronouns and vague or non-quantitative references. Students will frequently say that they "are approximating the hole" or "the point," but prompt them to be precise about describing the quantities. They need to clearly say that they want the value of the *y*-coordinate of the hole in the graph when $x = 5$.

c. Describe what you will use for **approximations**. Write a description of your answer using algebraic notation (for example, function notation, variables, formulas, etc.)

   If students don't see this quickly, have them use the trace feature on their calculator and note the *x* and *y* readouts. Make sure that they notice that when $x = 5$, the *y* coordinate is blank or says "error." But when tracing to that point, they will also likely notice that the *y*-values were changing. If they don't make the connection, ask what the meanings of those numbers are as they trace back and forth. They should express the answer using function notation, e.g., "the height of the hole is approximately $f(x)$ for values of *x* near but not equal to 5."

The Limit of a function at a point

The graph of $f(x) = \dfrac{\sqrt[3]{x+3} - 2}{x-5}$ has a hole. Your task is to determine the location of this hole using approximation techniques (no fancy limit computations allowed).

Class Instructions: Work with your group on the problem assigned to you. We encourage you to collaborate both in and out of class, but you must write up your responses *individually*.

1. Find an approximation to the height of the hole in your function (write out the approximation with several decimal places). Is this an **underestimate** or **overestimate**? Explain how you know. Find both an underestimate and an overestimate.

   Looking at the graph, someone in each group should quickly observe that since the function is decreasing around $x = 5$, values of x to the immediate left of 5 will generate overestimates and values of x to the right of 5 will generate underestimates. It is ok for them to note that f is decreasing near $x = 5$ simply based on looking at the graph on their calculator. Make sure to have other members of the groups repeat the reasoning to ensure they can also clearly articulate the justification. If some group members are not following the process, ask someone to clearly explain what they DID to compute the approximation. They may be tempted to average their approximations to get something "better" but ask them to just use the original function values since it is easy to see and explain why they are underestimates or overestimates. Pedagogically, this also keeps them engaged more directly in the structure of the limit $\lim\limits_{x\to 5} f(x)$.

2. Redraw your graph at a good scale to clearly illustrate how you can approximate the height of the hole. Label the unknown height and the approximation.

   Make sure that students' labels of the specific points on the graph are clear. Specifically, many are likely to simply put something like $f(5)$ near the point without any indication that this is the value of the height above the x-axis. Also make sure that they label both coordinates. Drawing in segments from the points to each axis, can help clarify this. Without this clarification, students are likely to think of the algebraic expressions as mere labels for the points, not as quantities.

3. Illustrate the **error** for your two approximations on your graph. Explain why you can't determine the numerical values of these errors. What is an algebraic representation for the error in your approximations?

   Students should draw a vertical segment from the height of the hole to the height of the $f(x)$ values used to generate approximations. They should represent this algebraically as $|y - f(x)|$. Push for the absolute values noting that for the size of the error, we are only saying how far off we are, not a direction.

Students are likely to initially apply a variety of different interpretations for the term "error," including ideas like % error, mistake, the calculator's response when trying to evaluate $f(5)$, etc. It is helpful to elicit these various interpretations as valid, but not what we mean in this course. Then focus them on the idea of how far off the approximation is from the actual value. This often works most efficiently as a brief whole-class discussion shortly after most groups have had time to start thinking about this question. It is also important for students to articulate that they will not get a numerical value for the error, otherwise they would have known the height of the hole, and there would be no need to approximate.

4. Use your underestimate and overestimate to find a **bound on the error** for these two approximations. Explain your work. Illustrate this error bound on your graph.

   Again, students are likely to have a variety of different initial interpretations for "bound on the error" and it is helpful to elicit these, then focus them on the idea of a small positive number that you know is larger than the error. Sometimes terms like "tolerance" come up from students and can be helpful. Motivate the need to think about a bound on the error by pointing out that if you don't know such a bound then your approximation is completely useless – you could be way off! Similarly, note that in different situations different sizes of error bounds may be appropriate, for example, NASA may need a smaller error bound for important variables when landing a rocket on Mars than you might need when deciding what time to leave your house to get to a party on time.

   Students will regularly confuse error and error bound, at least verbally if not conceptually. Constantly asking them which one they mean is helpful to force them to keep the two distinct and to use precise language. It is also helpful to ask them to articulate the relationship between the two, i.e., "the error is smaller than the error bound," and to have them write this algebraically, e.g.,
   "$|y - f(x)| < \text{EB}$." You may also ask them to use ε for the error bound.

5. List three fairly decent pairs of underestimates and overestimates (you can include the one you computed above). For each pair, give a bound for the error and use this to determine a range of possible values for the actual y-value of the hole in a table with headers as shown.

Underestimate	**Overestimate**	**Error Bound**	**Range of Possible Values**

Students may need help understanding the meaning of the "range of possible values." You can ask them, "Given this underestimate and this overestimate, what could the height of the hole be?" They are also likely to be sloppy and confuse this range either verbally or conceptually with error bound. They are very similar, but help them see that the error bound is a number (hopefully small) and the range of possible values is an interval of numbers (around the unknown y).

Students may require time to figure out how to generate an error bound. Having them return to the relative sizes of the error and error bound illustrated on the graph with an underestimate and overestimate pair clearly labeled can help spur this realization.

Do not let students use the average of an overestimate and underestimate. The point here is to keep the approximations equal to the argument of the limit, $f(x)$ in this case. You can also justify this by pointing out that if they use the average, they no longer know if it is an underestimate or overestimate (unless they bring in a concavity argument).

6. Find an approximation with error smaller than 1×10^{-5}. Then describe as best as you can ***all*** of the x-values you could use to get approximations that would have an error smaller than this error bound.

   Here, they should proceed by guess-and-check, using the method from Question 5 to determine error bounds. If they happened to have used an underestimate-overestimate pair in Question 5 that are within the desired accuracy, ask them to find something accurate to within a smaller error bound that you choose to force them to perform another computation. This experience should reinforce the idea that no matter what error bound is chosen, they can find a suitable approximation, $f(x)$, that they know is within that degree of accuracy, and that this happening is not just accidental.

7. For any pre-determined error bound, can you find an approximation with error smaller than that bound? Explain in detail how you know.

   This question explicitly generalizes the idea from Question 6, that this can always be done. They really only have the guess-and-check strategy at their disposal, but that will still allow them to say there is a way to do it. This question is NOT asking for students to give some sort of formal ε-δ definition. Detail of their guess-and-check procedure is appropriate.

Concluding discussion:

Review the approximation language and their multiple representations to recap the introduction of the approximation terminology and limit of a function at a point. In particular:

- Make sure students understand why the unknown cannot be computed directly, creating a need for approximations. There is a hole in the graph where we can't compute f because $f(5)$ is undefined.
- Students struggle with clearly distinguishing error from error bound. You can use the table to again point out the relationship $|f(x)-h|<\varepsilon$.

- Students struggle with distinguishing between different roles of error bound depending on the questions being asked. Help students see how drawing the error bounds for questions 4 & 5 (centered at the approximation) is different from drawing the error bounds for questions 6 & 7 (centered at the unknown value).
- The limit statement (last row of table) can be introduced as a statement that captures the entirety of this table.

Instructors should note that one activity is not enough to expect students to be using approximation terminology accurately. The new situations presented in subsequent labs will continue to challenge students to correctly identify what they are approximating, why they are approximating, and approximations. Expect students to continue to be imprecise with their use of error and error bound. Over several activities, students will become more precise with their language and more effective at successfully engaging these challenging problems. Eventually this conceptual structure will guide their reasoning and investigation into new topics defined in terms of limits.