LESSON TITLE: The Number *e*

OVERVIEW: This lesson serves as an introduction to the number *e* and the application of continuously compounded interest. Students begin by exploring the expression $\left(1+\frac{1}{n}\right)^n$ for large values of *n*, and they discover that the expression approaches a number (2.71828...) as *n* approaches infinity. For college algebra students, this also serves as an introduction to the idea of a "limit." Students then explore the relationship between compound interest and continuously compounded interest.

PREREQUESITE IDEAS AND SKILLS:

- Students should understand that increasing an amount by a percentage corresponds to a multiplication (e.g., increasing a number by 4% is equivalent to multiplying by 1.04)
- Basic calculator skills are required (including correct usage of parentheses, etc.)

MATERIALS NEEDED TO CARRY OUT THE LESSON

- Activity Worksheet
- Calculator
- Access to Desmos (optional)

CONCEPTS TO BE LEARNED/APPLIED

- Students will understand the definition of the number *e*, including recognizing the important covariational relationships between the quantities involved in the definition (i.e., the *size* of the factors vs. the *number* of factors).
- Students will make predictions about how the value of a certain expression changes as a variable in the expression approaches infinity (thus providing an introduction to the idea of a limit) and understand how this relates to a horizontal asymptote on a graph.
- Students will understand the concept of compound interest and make computations related to it, in particular understanding the covariational relationships among the number of compounding periods, the size of the increments of interest in each period, and the final account balance.
- Students will understand the relationship between compound interest and continuously compounded interest, recognizing the latter as a limiting case of the former.

INSTRUCTIONAL PLAN

The activity is divided into two parts:

- Part 1: the number *e*
- Part 2: compound interest (and continuously compounded interest)

It may not be possible to complete both parts in one class period, so a possible lesson plan is to cover Part 1 during one class period. Part 2 can then be completed in a future class period (either the next day of class, or at a future point in the semester). In fact, if the instructor simply wants to introduce the number *e* prior to teaching about logarithms and is not concerned with compound interest, then Part 2 may be skipped.

The activity begins with students making comparisons between various expressions involving exponents. Students then evaluate the expression $\left(1+\frac{1}{n}\right)^n$ for large values of *n*, and they see that as *n* becomes larger, two things are happening simultaneously: the number of factors is increasing, but the factors themselves are getting closer to 1. These counteract each other so that it is not clear whether we should expect, for example, $(1.000001)^{1{,}000{,}000}$ to be: a very large number, a number that is very close to 1, or something in-between.

Instructors should emphasize here that the "rate" at which the factors approach 1 compared with the "rate" at which the number of factors increases is important: $(1.00000001)^{1000}$ will be very close to 1, whereas $(1.01)^{1{,}000{,}000}$ will be extremely large, but $(1.001)^{1000}$ will be close to *e*.

Part 2 of the activity serves as an introduction to both compound interest as well as continuously compounded interest. If students have not yet been introduced to compound interest, the instructor should point out that for compound interest, the interest is applied not merely to the original amount of money, but also to the interest that has been gained. "The interest gains interest."

It is important for instructors to emphasize the important covariational relationships between: (i) the number of compounding periods, (ii) the size of the increments of interest in each period, and (iii) the final account balance at the end of the year. As the number of compounding periods increases, the increments of interest in each period become smaller, and the final account balance becomes larger (but by a possibly smaller amount than might be anticipated, especially as *n* approaches infinity). The timelines are given to provide a visualization of these relationships.

The goal is for students to not merely plug numbers thoughtlessly into a calculator, but to understand the nature of the quantities used in the computations. Therefore, it is vital that instructors emphasize the questions asked throughout the activity.

MIP COMPONENTS OF INQUIRY

This section outlines how the activity will meet the Mathematical Inquiry Project (MIP) criteria for active learning, meaningful applications, and academic success skills.

<u>Active Learning</u>: This activity will engage students in active learning as they discover the number *e*. Students are not merely given the definition of *e*, but actively explore how the number arises as a limit. Similarly, for compound interest, students create a formula for compound interest and expand that knowledge to understand continuously compounded interest as a limiting value that involves *e*. Students select activities in several places, including when they select among various scenarios involving compound interest, when they select large numbers for *n* to plug into $\left(1+\frac{1}{n}\right)^n$, and when they select among various expressions involving exponents to decide which is the largest and smallest.

<u>Meaningful Applications</u>: This activity will emphasize meaningful applications involving the number *e* and compound interest. Students use covariational reasoning when they consider how both the "progressively smaller size of the factors" and "increasingly large number of factors" affect the final output of the computations. They will also make claims as to whether they can become a billionaire if

the interest is compounded enough times. Lastly, this activity introduces the idea of a limit to students in a pre-Calculus class in preparation for a more formal development of the idea in a future class.

Academic Success Skills: This activity will allow students to make mathematical discoveries themselves as opposed to simply being given formulas. Moreover, students will enhance their financial literacy through a renewed understanding of compound and continuously compounded interest.

Why does the number *e* appear in the formula for continuously compounded interest?

The student (and the instructor) may wonder exactly how the compound interest formula gives rise to the continuously compounded interest formula as *n* approaches infinity. In other words, how can we show that:

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\, e^{rt}$$

In a college algebra class, it would be quite difficult for a student to verify this, especially without having facility with limits and limit notation. For the instructor, however, who is teaching more advanced students (i.e., calculus students), the following calculation can be shown.

Fix *P*, *r*, and *t*, and compute:

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt}$$

$$= P\left(\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}\right)^{t}$$

(Here, we make the change of variables $k = \frac{n}{r}$)

$$= P\left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{kr}\right)^{t}$$

$$= P\left(\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k}\right)^{rt}$$

$$= P\, e^{rt}$$

Notice how the definition of *e* is used in the final step.

At the level of college algebra, it is sufficient for a student to notice that the calculations they performed when computing compound interest with large values of *n* are *similar to* the calculations that they performed in the context of the number *e*.