Name: ______________________________

Instructional Activity: The number *e*

This activity consists of two parts. In the first part, we explore the number *e*, which (like the number π) is one of the most important constants in mathematics. In the second part, we investigate compound interest and its relationship to the number *e*.

PART 1: The Number *e*

Before we begin:

When a positive number (such as 7) is multiplied by a number that is slightly larger than 1 (such as 1.00024), how will the result compare to the original number? (Will it be bigger or smaller than the original number? By a lot, or not very much?)

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1. Without using a calculator, select which of the following you think will be the largest and which will be the smallest. Explain your reasoning below.

$$1.01^{100}$$

$$1.01^{1000}$$

$$1.001^{100}$$

2. Use a calculator to evaluate the expression $\left(1+\frac{1}{n}\right)^n$ for "large" values of n (such as $n = 10$, $n = 100$, and $n = 100{,}000$).

(a) $n = 10$

$\left(1+\frac{1}{10}\right)^{10} = (1.1)(1.1)(1.1)\ldots(1.1)$ = ____________________

(10 factors total)

Notice that we begin with the number 1.1, and we keep multiplying by 1.1 until we have a total of 10 factors.

(b) $n = 100$

$\left(1 + \frac{1}{100}\right)^{100} = (1.01)(1.01)(1.01) \ldots (1.01) =$ ____________________________

(100 factors total)

Each time a number is multiplied by 1.01, the result is a small amount larger (1% larger). Yet in this calculation, we are performing a large number of multiplications (a total of 100 factors).

Before answering part (c), make a guess how large the result will be for $n = 100{,}000$. Do you think it be: (i) extremely large, (ii) something very close to 1, or (iii) something in-between? Explain your reasoning. (Note: there will be a <u>huge number of factors</u>, but each factor will be <u>very close to 1</u>.)

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(c) $n = 100{,}000$

$\left(1 + \frac{1}{100{,}000}\right)^{100{,}000} = (1.00001)(1.00001) \ldots (1.00001) =$ __________________________

(100,000 factors total)

(d) Now select some of your own (even larger) values for *n*.

$n =$ ____________________ $\left(1 + \frac{1}{n}\right)^{n} =$ ___________________

$n =$ ____________________ $\left(1 + \frac{1}{n}\right)^{n} =$ ___________________

3. What happens to the expression $\left(1 + \frac{1}{n}\right)^{n}$ as n gets greater and greater?

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We <u>define</u> the number e to be the "limit" of the expression $\left(1 + \frac{1}{n}\right)^{n}$ as n "approaches infinity."

$$e \approx 2.718281828459045$$

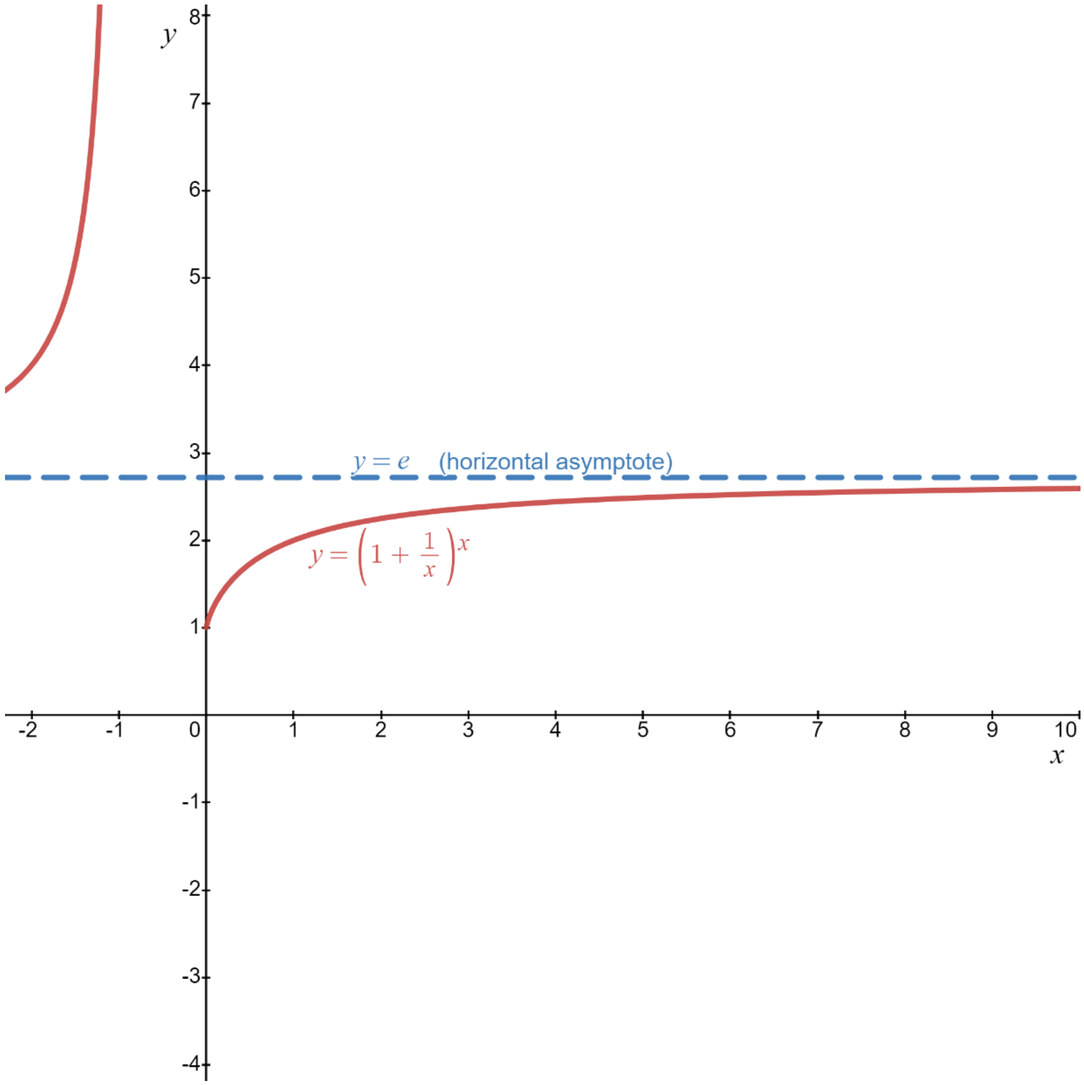
In the notation of calculus, we would write:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n}$$

$$\approx 2.718281828459045$$

Side note: It turns out that e, like the number π, is an *irrational number*. This means that it cannot be written as a ratio of integers.

The graph of $y = \left(1 + \frac{1}{x}\right)^x$ is below:



What happens to the *y*-values on the graph as the *x*-values get larger and larger?

Notice that the horizontal line $y = e$ (i.e., the dotted line in the picture above) is a horizontal asymptote of the graph of $y = \left(1 + \frac{1}{x}\right)^x$. This graph can be found on Desmos at the link: https://www.desmos.com/calculator/hgbeetmbe3

The number e has many applications beyond pure mathematics, including in engineering, physics, economics, and continuously compounded interest. See here: https://www.nature.com/articles/s41567-019-0655-9

PART 2: Compound Interest (and its relationship to the number *e*)

Question: Suppose you are about to deposit \$1000 into an account that earns interest. (Assume there are no bank fees, and that no additional money will be deposited or withdrawn.) Select the scenario (below) in which the account will have the most money at the end of one year. Do not use a calculator. Explain your reasoning below.

Scenario A: The money grows by 8% in one year.

Scenario B: The money grows by 4% in the first 6 months, and then grows by 4% in the next 6 months.

Scenario C: The money grows by 2% in the first 3 months, by 2% in the next 3 months, by 2% in the next 3 months, and by 2% in the final 3 months.

[Note: The amount of money at the end of one year will NOT be the same for each of these scenarios.]

Fill in the boxes on the timelines below to determine how much money is in the bank account at the end of 1 year if the interest is compounded 1 time, 2 times, and 4 times. **Each timeline (below) represents 1 year.**

Compounding just 1 time (i.e., simple interest):

8%

\$1000	
Initial amount	after 1 year

Compounding 2 times:

4%		4%	

Initial amount	after 6 months	after 1 year
$1000	______	______

Note: Your answer in the final box (above) should NOT be $1080. It should be a little more than that. To get your answer in the middle box, you multiplied the amount in the first box (i.e., $1000) by 1.04. To get the answer in the final box, you then need to multiply the amount in the <u>middle box</u> by 1.04.

Explain why the balance after 1 year is more when compounding twice than compounding once.

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Compounding 4 times:

2 %	2%	2%	2%

Initial amount	after 3 months	after 6 months	after 9 months	after 1 year
$1000	______	______	______	______

Why did we use 2% in the above calculation? The interest rate is 8%, and since we are compounding 4 times per year, we divide 8% by 4 to get 2%.

Notice that the account balance, *A*, after one year was ultimately found by taking $1000 and multiplying repeatedly (4 times) by 1.02:

$$A = 1000(1.02)(1.02)(1.02)(1.02) = 1000(1.02)^4 \approx 1082.43216$$

But since $1.02 = (1 + .02) = 1 + \frac{.08}{4}$, the account balance after 1 year was found as follows:

$$A = 1000\left(1 + \frac{.08}{4}\right)^4$$

$$\approx 1082.43216$$

Write an equation (like the one above) that will give the balance, A, at the end of one year if the interest is compounded monthly (12 times per year), assuming that \$1000 is invested at 8% interest:

$A =$

$\approx$ ______________

What do you expect will happen if the interest is compounded daily (365 times per year)? Or one billion times per year? In the latter case, would the account holder become extremely wealthy by the end of the year, like Mark Zuckerberg or Elon Musk?

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It turns out that if the interest were compounded 1 billion times per year, the account balance at the end of the year would be:

$$1000\left(1+\frac{0.08}{1{,}000{,}000{,}000}\right)^{1{,}000{,}000{,}000} = 1083.28707\ldots \approx \$1083.29.$$

Notice that the calculation on the left-hand side looks very similar (if we ignore the 1000) to our calculations involving the number e. In fact, when the number of times the interest is compounded approaches infinity, we get something called "continuously compounded interest," and the number e is used when calculating it!

Continuously compounded interest: (for \$1000 at 8% annual interest, compounded continuously for 1 year)

$$1000\, e^{0.08} = 1083.28707\ \ldots \approx \$1083.29$$

FORMULAS:

Compound Interest: $$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = final amount

P = initial amount

r = interest rate (expressed as a decimal)

n = number of compound periods per year

t = time, in years

Note: In all our calculations, we had $P = 1000$ and $r = .08$. We also had $t = 1$, because we assumed the time was 1 year. But we could look at longer time periods (for example, $t = 5$ for 5 years).

Continuously Compounded Interest: $$A = P\, e^{rt}$$

A = final amount

P = initial amount

r = interest rate (expressed as a decimal)

t = time, in years