Activity 1: Hoverboards and Rates of Change

Hoverboards: Rates of Change, and Limiting Values – Instructor Guide

This instructor guide begins with an introduction, prerequisite ideas, a conceptual analysis, a learning trajectory, and a description of how the tasks exemplify the three components of mathematical inquiry. The activity itself is then presented along with some answers and specific instructional notes for parts of the tasks. A blank, student copy of the activity follows those pages.

Learning trajectory

This activity is designed to support students in developing quantitative and covariational notions of rates of change and their relationship to long-term behavior of the function. Specifically:

- We want students to understand average rate of change as a ratio of the change in a dependent quantity for a fixed change in the independent quantity.
- We want students to understand that the average rate of change can be used to estimate unknown values. Because these are estimations, one must be careful and understand that the actual value may differ from the estimate.
- We want students to understand a limiting value as the value for the dependent variable for which the average rate of change approaches 0. This is different from thinking of a limiting value as a graph "leveling off." Perceiving a limiting value as an attribute of the shape of a function's graph provides little insight into how the input and output quantities change together. Moreover, it can lead students to believe that the quantities "stop changing" when, in fact, the output becomes near-constant as the input continues to vary.

These tasks rely on the following prerequisite ideas and skills:

- Understanding a function as a description of how two quantities change in relation to each other.
- Understanding function notation.
- The ability to view and interpret functions given by tables.
- Knowing how to construct an "expanded table" (a format introduced in previous lessons).
- Knowing how to compute an average rate of change, and having a characterization of an average rate of change as the change in the output if the input changes by 1 unit.
- Knowing how to convert a percentage to a decimal and understanding that percentage change is multiplicative.

Phase 1: Identifying changes in the dependent quantity for a fixed change in the independent quantity.

- Students conceptualize $f(x)$ as a relationship in which f depends on x.
- Students conceptualize $f(x_i)$ as the value of f at a particular value of x, and understand that the unit is the unit of the dependent variable.

- Students conceptualize $f(x_{i+1}) - f(x_i)$ as the change in f over the interval $[x_i, x_{i+1}]$, and understand that the unit is the unit of the dependent variable.
- Students conceptualize $x_{i+1} - x_i$ as the change from x_i to x_{i+1}, and that the unit is the unit of the independent variable.
- Students recognize the expression $[f(x_{i+1}) - f(x_i)] / (x_{i+1} - x_i)$ as the change in f over the interval $[x_i, x_{i+1}]$, and view f and x as changing simultaneously. Students see that the unit is hence "unit of the dependent variable" per "unit of the independent variable".
- Students obtain a value for $[f(x_{i+1}) - f(x_i)] / (x_{i+1} - x_i)$ and understand it as the change in f for a 1-unit change in x.

Phase 2: Using the average rate of change to make predictions.

- Students iterate the value of $[f(x_{i+1}) - f(x_i)] / (x_{i+1} - x_i)$ for integer changes in x to make estimations.
- Students perform multiple computations predicting the value of the dependent variable, and predicting the value of the independent variable for a given value of the dependent variable, to help them abstract the common structure of "prediction = base value + AROC*(change in independent variable)

Phase 3: Understanding a limiting value as the value of the dependent variable when the average rate of change approaches 0.

- Students, given a tabular representation of a function with a limiting value, compute the changes in the output and the average rates of change for the function for successive, fixed intervals of the independent variable.
- Students observe a decreasing change in the output and decreasing average rate of change.
- Students observe that the change in the output and the average rate of change decrease to near-0 (or 0).

Active learning: One example of students working to resolve a problematic situation occurs in problem 1b, when students are asked to interpret function notation without using numbers. Students often struggle to do this, and asking them to interpret the notation without evaluating the function helps them abstract a common structure for f(b) – f(a) as the change in f from a to b and b – a as a change from a to b. This later helps them build a quantitative, covariational understanding of average rate of change.

As a second example of requiring students to select, perform, and evaluate actions whose structures are equivalent to the concepts to be learned, problems 1c and 1f require students to realize they need to employ the average rate of change to make a prediction. 1c has students predict an output while 1f requires them to reverse the process to predict an input; this reversibility helps students abstract the common structure "prediction = base + AROC*(change in input)".

As a third example, in problem 2b, students compute changes and average rates of change for a function with a limiting value, then reflect on the answers to build a conception of a limiting value as that which occurs when the changes in the dependent variable and average rate of change approach 0.

Academic success skills: These tasks, meant to be done in small groups, engage students in productive struggle (e.g., it is likely that students will struggle to interpret function notation in practical terms without using numbers). There are multiple instances for students to practice particular skills, which allows students to feel successful and subsequently build their images of themselves as capable doers of mathematics. Other tasks (e.g., 1c, 1f) engage students in problem-solving by asking non-standard questions (e.g., 1f).

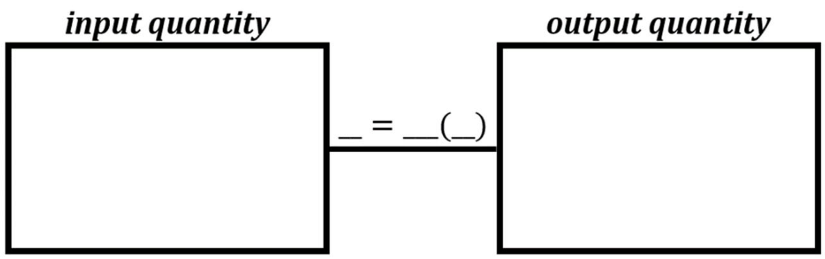
It is possible that being able to work collaboratively could lower students' mathematics anxiety by creating opportunities for students to succeed in mathematics, with support.

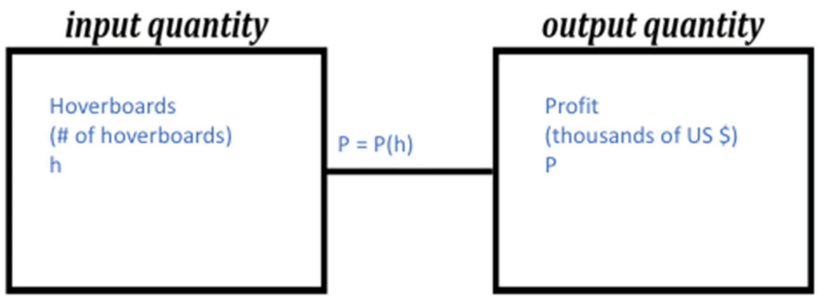
Meaningful applications: The problems described in 'active learning' are examples of applications that support students in identifying relationships such as changes in quantities. Problem 1d engages students in making and justifying a claim about how to make predictions using the average rate of change. Problems 1c and 1f help students abstract a common structure by asking them to perform a process in one problem, then reverse the process in another.

Problem 1. Suppose you are part of a group running a small business that makes hoverboards (not the crappy kind that doesn't actually hover, but the kind from Back to the Future). You hire an analyst to help you determine how much money you can make (i.e., profit) from producing a certain number of hoverboards in a given year. The analyst gives you and your team the following function (in the form of a table), which describes how profit depends on the number of hoverboards produced:

# of hoverboards, *h*	Profit, *P* (in thousands of US dollars)
20	-3287.1
40	-569.1
60	1348.9
80	2466.9
100	2784.9

(a) Identify the quantities involved and the units in which they are measured. Define a variable to represent each quantity, and classify each as an input or output quantity. Finally, express the relationship in function notation.





(b) Provide an interpretation of what each of the following expressions means. Include in your interpretation what the units for each expression are. Do not respond with the numerical value of these expressions.

i. $P(80\ HBs)$

$P(80)$ is the profit, in thousands of US dollars, from producing 80 hoverboards. Emphasizing that students should not respond with a numerical value helps students construct a meaning for P(80) as the completed process of having evaluated the function for an input of 80 hoverboards and obtained a profit value. Students often see notation like this as specifying "do something", and we want them to build a meaning for it in which they can envision it as the output for a particular input (which is part of the value of function notation).

ii. $P(100\ HBs)$

$P(100)$ is the profit, in thousands of US dollars, from producing 100 hoverboards.

iii. $100\ HBs - 80\ HBs$

100 HBs – 80 HBs is the change in production level from 80 hoverboards to 100 hoverboards. Help students interpret this as a "change in" production level rather than the phrase "the difference in…". The goal is to help them see the denominator of the average rate of change formula as a change in inputs, so the phrase "change in" is useful here.

iv. $P(100\ HBs) - P(80\ HBs)$

P(100 HBs) – P(80 HBs) is the change in profit, in thousands of dollars, from a production level of 80 hoverboards to a production level of 100 hoverboards. Like iii, help students use the phrase "change in…" and ensure that they attend to these as profits at specific production levels. This helps them conceptualize the quantities as covarying.

v. $\frac{P(100\ HBs)-P(80\ HBs)}{100\ HBs-80\ HBs}$

This is the average rate of change in profit as production changes from 80 to 100 hoverboards, in thousands of dollars per hoverboard.
If students struggle with this, direct them to their answers for iii and iv.

(c) We have discussed how useful tables can be – after all, they provide us with an easy way to organize lots of important information – but one of their downsides is that they leave gaps. For example, what is the profit corresponding to a production level of 93 hoverboards? 97 hoverboards? We don't know (at least not immediately). The average rate of change provides one way that we can fill in these gaps in the table by *estimating*. First, calculate the average rate of change for the production interval between 80 and 100 HBs. Then devise a way to *estimate* the profit corresponding to production levels of (1) 93 hoverboards, and (2) 97 hoverboards.

The average rate of change from 80 to 100 HBs is
$\frac{P(100\ HBs)-P(80\ HBs)}{100\ HBs-80\ HBs} = \frac{2784.9-2466.9}{100-80} = \frac{318}{20} = 15.9$ thousand dollars per hoverboard
The prediction for 93 hoverboards is 2466.9 + 13(15.9) = 2673.6 thousand dollars.
The prediction for 97 hoverboards is 2466.9 + 17(15.9) = 2737.2 thousand dollars.
Having students devise a way to use the ARoC to estimate the profit engages students in problem-solving and honors multiple solution paths, both of which would be circumvented by showing students how to use the ARoC to make predictions.
If students struggle, ask them to estimate the profit corresponding to 91 hoverboards, then 92 hoverboards. Another hint is to have students focus on the units for the average rate of change.

(d) Explain your method. (Meaning: write a step-by-step list of instructions called "how to make estimations using the average rate of change.")

Answers will vary
Having students write the instructions engages them in reflection about their method, which can aid in building understanding and abstracting the general structure of the computation (to wit, starting value + (desired change in independent variable)*AROC)

(e) Explain why the values you have proposed for $P(93\ HBs)$ and $P(97\ HBs)$ are *estimations* (i.e. educated guesses) and not necessarily the actual values for $P(93\ HBs)$ and $P(97\ HBs)$.

Answers will vary

This helps students unpack an important idea about ARoC, namely, that it assumes a constant rate of change. (This can be leveraged later when studying linearity to characterize linear functions as those with a constant ARoC.) In actuality, the profit may not have a constant rate of change and thus might have a different value than the predictions.

(f) A *break-even* point is a production level that corresponds to a profit of \$0 (i.e. you are bringing in exactly as much money as you are spending). Use your method for estimations with the average rate of change to estimate the how many hoverboards you need to produce to break even.

The break-even point occurs between 40 and 60 hoverboards, The ARoC is $\frac{P(60\ HBs)-P(40\ HBs)}{60\ HBs-40\ HBs}=\frac{1348.9-(-569.1)}{60-40}=\frac{1918}{20}=$ 95.9 thousand dollars per hoverboard.
Then we want
0 = -569.1 + 95.9h (where h is number of hoverboards past 40 hoverboards)
5.9 = h, so about 46 hoverboards
If students use the ARoC from the previous problem, they will get about 75 hoverboards, which is not correct because we know profit is already positive at 60 hoverboards. If they do this, it is an opportunity to point out that the ARoC is different for different intervals.
This problem provides students with an opportunity to calculate another ARoC; they were given the setup to compute an ARoC in part (a)v above, but in this problem they must attend to that structure to compute a different ARoC. Additionally, while previous problems asked students to predict the profit (an output), this problem reverses the process and directs students to find the input.

(g) Expand the original table given at the beginning of this problem (to include changes in both quantities as well as the average rate of change). Use this additional information to make a prediction about what will happen to profits if production levels continue to increase past 100 HBs.

An expanded table, something that would have been introduced previously, looks like this:

Δh (measured in hoverboards)	$h =$ $\#\ of$ hoverboards	$P =$ profit (in thousands of dollars)	ΔP (measured in thousands of US dollars)
----------------------	20	-3287.1	----------------------
20			2718
20	40	-569.1	1918
	60	1348.9	
20			1118
	80	2466.9	
20			318

	100	2784.9	

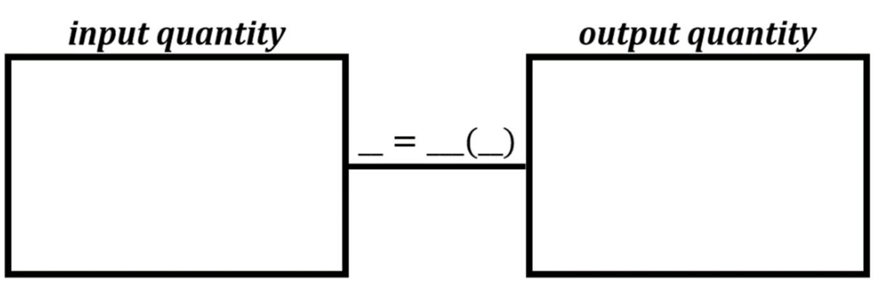
The average rate of change is decreasing, so past 100 hoverboards, profit may decrease (if average rate of change becomes negative).
Attending to units ("measured in") helps students with quantification. Computing Δh supports students in understanding the importance of a fixed change in the independent variable as necessary to compare changes in the dependent variable. Computing this, and ΔP, later support students in a quantitative understanding of concavity, which is more powerful than focusing on perceptual features such as "up like a cup or down like a frown." These computations can help students understand what it means to "increase at a decreasing rate".

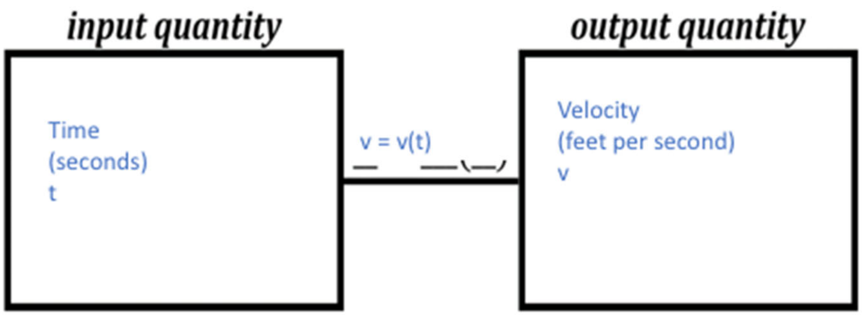
Problem 2. For some unknowable reason, you are a fan of sky diving. (Why anyone would jump out of an airplane and hurtle towards the earth at literally breakneck speeds is beyond my comprehension.) Ten seconds after you jump, your velocity is 147 feet per second. Consider your velocity v (in feet per second) as a function of the number of seconds t since you jumped.

Time t (seconds)	Velocity v (feet per second)
0	0
10	147

(a) Identify the quantities involved and the units in which they are measured. Define a variable to represent each quantity, and classify each as an input or output quantity. Finally, express the relationship in function notation.

We start all problems like this because it engages students in the mental activity of making sense of quantities and their relationships.





(b) Show that using the method of estimation with the average rate of change to predict your speed after 60 seconds results in a prediction of 882 feet per second (roughly 600 mph).

> By giving the answer, students must focus on the process. This is another opportunity to abstract the common structure of base amount + (desired change in input)*ARoC.

(a) It turns out, thankfully, that you don't end up falling at 882 ft/s (600 mph) because of something called wind resistance. Your *actual* velocity after 60 seconds is 176 ft/s. (This is still terrifying, but I would take some small comfort in the fact that, if I were to plummet to my death, at least I wouldn't do it as quickly.) In your own words, write a caution/warning statement here for using the method of estimating with the average rate of change:

(b) The *actual* velocities (including the 176 ft/s mentioned in the previous part) are given in the following table.

Δt (measured in ________________ ________)	Time t (seconds)	Velocity v (feet per second)	Δv (measured in ________________ ________)	AROC: $\frac{\Delta v}{\Delta t}$ (measured in ________________ ________)
----------------------	0	0	----------------------	---------------------- -
	10	147		
	20	171		
	30	175		
	40	175.8		
	50	176		
----------------------	60	176	----------------------	

				--------------------- -

Δt (measured in seconds)	Time t (seconds)	Velocity v (feet per second)	Δv (measured in ft/sec)	AROC: $\frac{\Delta v}{\Delta t}$ (measured in ft/sec per second)
----------------------	0	0	----------------------	--------------------- -
10			147	14.7
	10	147		
10			124	12.4
	20	171		
10			4	0.4
	30	175		
10			0.8	0.08
	40	175.8		
10			0.2	0.02
	50	176		
10			0	0
----------------------	60	176	----------------------	--------------------- -

(c) Notice that, as time passes (i.e. as the input quantity increases), the output values v start to cluster around (approach) 176 ft/s. Explain what is happening to (1) the change in velocity Δv, and (2) the average rate of change as this is occurring. Next, read the definition of limiting value and fill in the blanks.

As the values cluster around 176 ft/sec, the change in velocity approaches 0 (velocity stops changing as time changes). The average rate of change decreases to 0.
This problem supports students in building a quantitative understanding of a limiting value.

Definition 1. A ***limiting value*** occurs when …

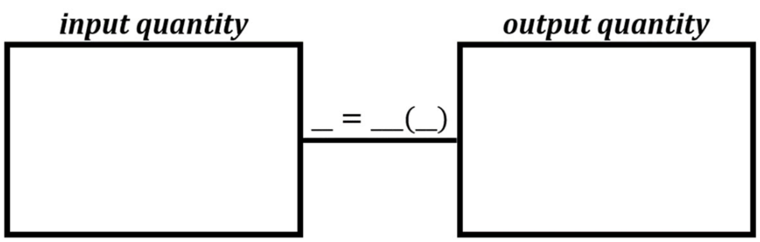
- … the *output values* get closer and closer to one specific output value as the input values increase. The limiting value is the specific output value that the others are clustering around (getting closer and closer to).
  - For example, the limiting value in the sky diving scenario is 176 ft/s).
- … the *changes in the output* get cluster around (get closer and closer to) ______ for successive, equal changes in the input.
- … the *average rate of change* clusters around (gets closer and closer to) _______ as the input values increase.

This is a quantitative characterization of a limiting value, which helps students understand what causes the perceptual "leveling off" in a graph.

Problem 4. Suppose that you leave town for a few weeks over the summer without returning your university library books – oops – and return for the fall semester to find that you owe $430 in fines. Since you don't have $430 to give them, you work out a payment plan with the library staff: at the end of each month, you will pay the library 20% of your remaining balance. Consider your remaining balance B as a function of months since you started the payment plan. The relationship between these quantities is described by the table below.

This problem helps students extend their learning to another context, providing an opportunity for them to reflect on learning and abstract common structures.

(a) Address the Big 5.



(b) Explain what the following expressions represent (at this point, do not respond with a numerical answer):

$$B(0 \text{ months})$$

$$B(4 \text{ months})$$

$$B(4 \text{ months}) - B(0 \text{ months})$$

$$\frac{B(4 \text{ months}) - B(0 \text{ months})}{4 \text{ months} - 0 \text{ months}}$$

(c) Show that the numerical value of $B(4 \text{ months}) - B(0 \text{ months})$ is $-\$253.87$. Then explain why it is *negative.*

(d) How much of the balance have you paid back after 4 months? Which of the above expressions does this quantity refer to?

(e) Show that the average rate of change between 0 months and 4 months is $-\$63.47$ per month. Then write a sentence explaining what this means. (It might be helpful to refer to the notes from Class 4, which include our summary of what the average rate of change is.)

(f) Expand the table. Start by (1) filling in the units for each of the three remaining columns, and (2) filling in the values you have already calculated in previous parts (including $-\$253.87$ and $-\$63.47$ per month).

Δm (measured in ____________)	Months since starting the payment plan, m	Remaining balance, B (in dollars)	ΔB (measured in ____________)	AROC: $\frac{\Delta B}{\Delta m}$ (measured in ____________)
----------------------	0	$430	----------------------	----------------------
	4	$176.13		
	8	$72.14		
	12	$29.54		
	16	$12.10		
	20	$4.95		
----------------------	24	$2.03	----------------------	----------------------

(g) Does this situation have a limiting value? Explain why or why not by explaining what is happening (as the values of the input quantity increase) to …

- … the values of the output quantity:
- … the changes in the output quantity:
- … the average rate of change: