Quantitative Reasoning about Rates of Change
College Algebra / Precalculus CoRD

The following four activities are intended to be used throughout a College Algebra or Precalculus course to develop and reinforce students' quantitative reasoning about rates of change. The activities first reinforce thinking about changes in quantities then encourage comparing those changes for two covarying quantities. The activities then focus on working with constant rates as a constant ratio between changes in two covarying quantities. We recommend spacing out implementation of the five activities in a course to give students opportunities to reflect on each one as they learn other material in the course. In this way, reasoning about covarying quantities, changes in those quantities, and rates of change may become a standard habit of mind for students. The contexts and problem-solving in these activities can provide students a general frame of reference for such reasoning.

Since we developed these resources for the Mathematical Inquiry Project, the activities are designed to reinforce the following components of inquiry as defined by the MIP:

Students engage in *active learning* when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.

Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.

Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.

Conceptual Analysis

Rate of change is fundamental to students' understanding of content throughout College Algebra and Precalculus. A key defining feature of linear functions is their constant rate of change. Students must be able to identify, represent, and work with constant rates throughout most linear contexts. Exponential functions are characterized by the fact that rate of change is proportional to the function value. This is true for average rates take over a consistent input interval as well as for instantaneous rate and is the primary feature that makes exponential functions foundational to modeling across the sciences. Analyzing and understanding the patterns of varying rates of change are essential for students' use and interpretation of other function classes.

Carlson et al. (2002) presented a framework detailing various levels of mental actions involved in acts of reasoning about two covarying quantities.

> **Mental Action 1 (MA1):** Coordinating the value of one variable with changes in the other.
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> **Mental Action 2 (MA2):** Coordinating the direction of change of one variable with changes in the other.
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> **Mental Action 3 (MA3)**: Coordinating the amount of change of one variable with changes in the other.
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> **Mental Action 4 (MA4):** Coordinating the average rate of change of the function with uniform increments of change in the input variable.
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> **Mental Action (MA5):** Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable.

Students often attempt to reason intuitively with mental actions MA4 and MA5, but are not able to call on the imagery of the underlying mental actions MA1-MA3 to support that reasoning. This often leads students to reason in ways that are purely figurative and not quantitative, thus leading to serious misconceptions. For example, students are likely to view steepness as an icon standing for "faster." They are thus unable to interpret rates when presented graphically with different scales, or even more, when presented in algebraic, numerical, or verbal representations. Another common error is to conflate features of a physical situation with features of the graph, for example assuming that the graph of a biker's speed riding up and over a hill must resemble the shape of that hill. Finally, the mental actions are somewhat hierarchical in that meaningful application depends on the understandings of the lower-numbered mental actions. This does not mean, however that the higher numbers are associated with more important or sophisticated reasoning. Experts often unpack and interpret their understanding about varying rates or seek better insights into a quantitative relationship by invoking the lower level actions.

Most prominently, students often struggle to unpack their reasoning in terms of MA3, for example by imagining successive equal increments of an input value for a function and the corresponding changes in the output value. Attending to qualities such as whether these changes in output increase, decrease, or remain constant provides students' powerful and intuitive ways to understand varying rates of change, and build more robust understandings of average rate and instantaneous rate represented in MA4 and MA5. This set of activities focuses students on identifying and describing patterns of covariation in terms of MA3 in order to develop this quantitative foundation for understanding rate of change.