Activity 3: Characterizing Covariation in Terms of Rate of Change

Purpose and Rationale: The purpose of this activity is to support students' ability to characterize the covariational relationship between a function's input and output quantities using the language of quantitative relationships, not visual features or figurative properties of graphs.

Researchers who have studied students' understanding and learning of functions, including their representations and applications, have called for algebra and precalculus courses to support students in viewing a function as a *process* (e.g., Breidenbach et al., 1992; Dubinsky & Harel, 1992) that relates two quantities as their values vary in tandem (Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008; Thompson & Carlson, 2017). A *process view* of function involves conceptualizing an expression as self-evaluating in the sense that it represents the result of its evaluation without having to perform concrete computations (Thompson, 1994c). This way of thinking enables students to anticipate the coordination of input and output values as they covary, and is thus foundational for constructing meaningful formulas and graphs when modeling relationships in applied contexts (Carlson et al., 2002; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008). A student with a process view of function can imagine running through a continuum of inputs and has no difficulty coordinating, reversing, or composing two function processes (Oehrtman, Carlson & Thompson, 2008).

Regarding the graphical representation of functions, Moore and Thompson (2015) and Moore (in press) introduced the theoretical construct *static shape thinking* to describe students' meanings for graphs grounded in perceptual features. A student who engages in static shape thinking makes associations between mathematical terms or inscriptions and visual properties of (or actions on) a graph as an object so that these perceptual associations *constitute* one's meaning for such terms and inscriptions (e.g., constant rate of change means "straightness"; exponential growth means "curving up"; quadratic means "U-shaped"; inverse functions mean "flip over the diagonal"). This is in contrast with *emergent shape thinking*, which involves conceiving figurative properties of a graph as having emerged from representing the simultaneous variation of quantities' measures in a coordinate system. A student who engages in emergent shape thinking interprets figurative aspects of a graph as representing properties of constrained covariation (Moore & Thompson, 2015, p. 786). Moore, Stevens, Paoletti, Hobson, and Lang (2019) document the problematic nature of students' static shape thinking and provide evidence of the affordances of emergent shape thinking for enabling students to accommodate novel figurative material and experiences.

Informed by this literature, the purpose of this activity is to support students' emergent shape thinking about functions represented graphically, and to describe emergent properties of graphs in the language of quantitative relationships, specifically average rate of change. It is important that prior to this activity, students be supported in constructing a conception of constant rate of change as a proportional relationship between corresponding changes in covarying quantities' measures. It is also important that students conceptualize average rate of change as the constant rate of change needed to change a specific amount in the output quantity for a specific amount of change in the input quantity. In particular, the average rate of change of a function f from $x = x_1$ to $x = x_2$ is the constant rate of change of a linear function g that has the same change in output as the function f over the interval $[x_1, x_2]$.

Instructions: Students will be grouped into pairs. One student in each pair will receive a copy of either Graph 1 or Graph 2. The other student in each pair will receive either Coordinate System A or Coordinate System B. The student with the Graph 1 or 2 will describe the covariation of the input and output quantities of the function graphed to their partner, who will attempt to sketch the function on the coordinate system they have been provided. *It is important that neither student show the other their handout while working on this activity*. Students will work collaboratively to complete remaining tasks.

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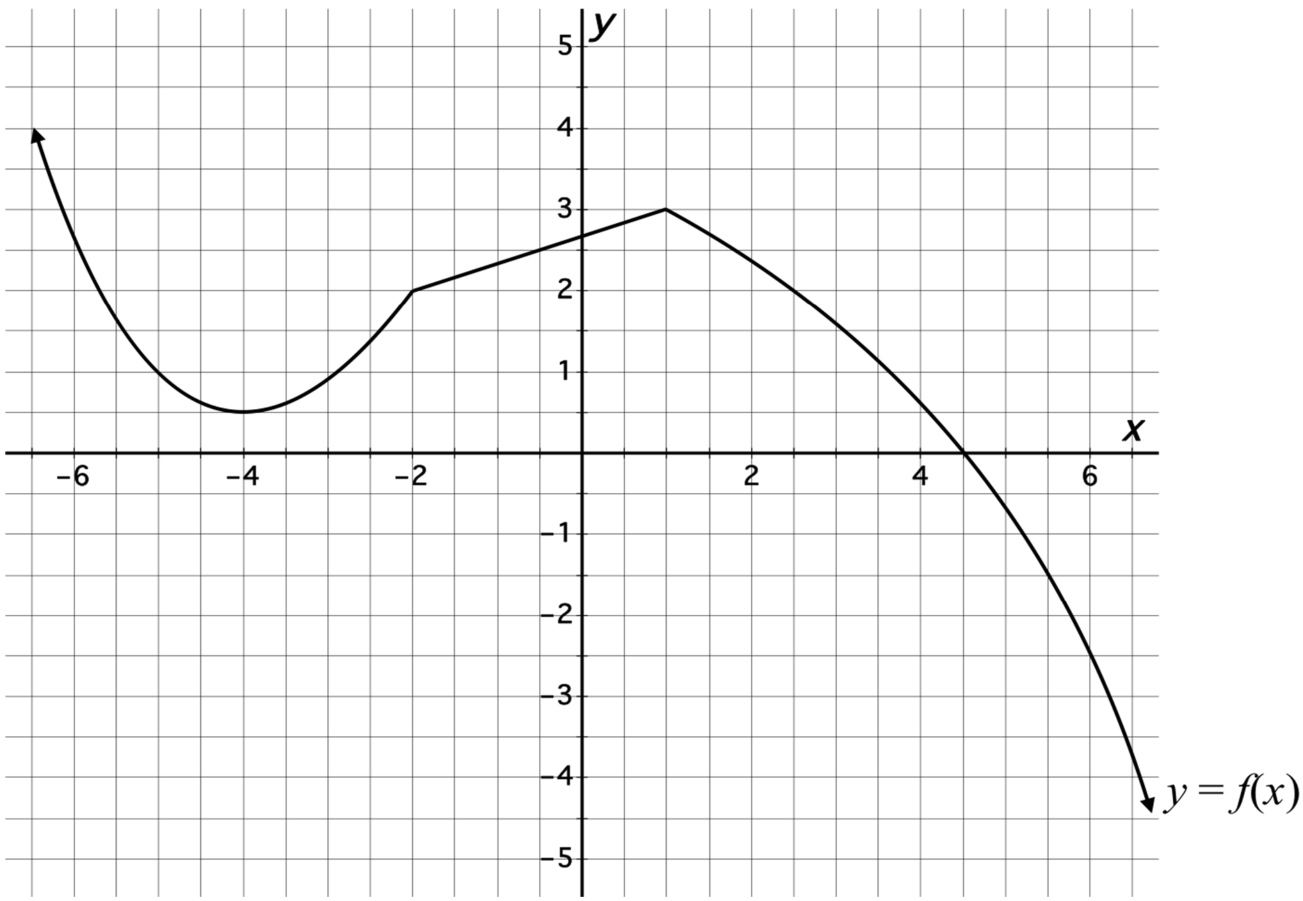
GRAPH 1

Do not show this graph to your partner. Describe the covariation of x and $f(x)$ to enable your partner to sketch a graph of the function. Do not describe figurative properties of the graph (i.e., "it goes up," "it is straight," "it curves down").



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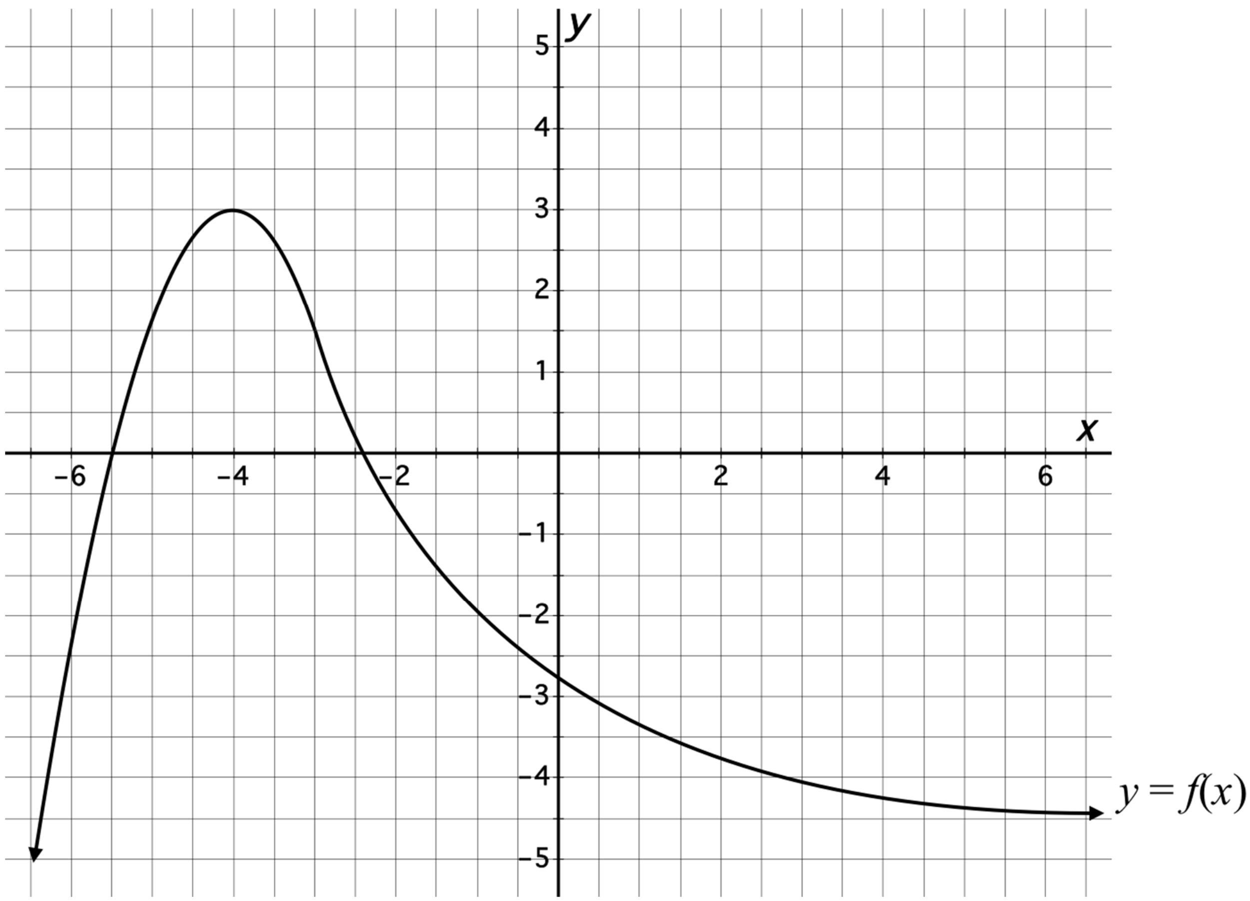
<u>GRAPH 2</u>

Do not show this graph to your partner. Describe the covariation of x and $f(x)$ to enable your partner to sketch a graph of the function. Do not describe figurative properties of the graph (i.e., "it goes up," "it is straight," "it curves down").



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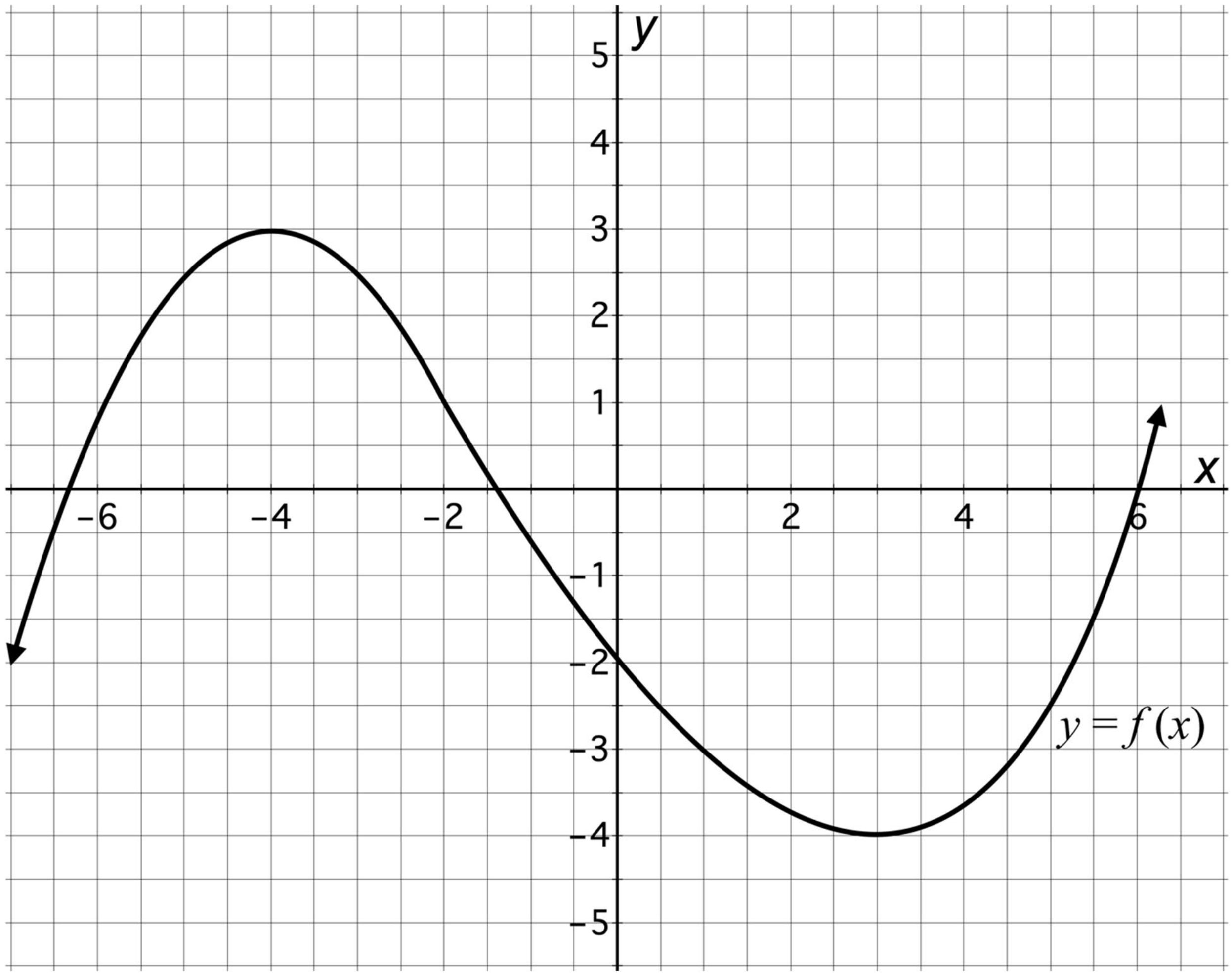
GRAPH 3

Do not show this graph to your partner. Describe the covariation of x and $f(x)$ to enable your partner to sketch a graph of the function. Do not describe figurative properties of the graph (i.e., "it goes up," "it is straight," "it curves down").



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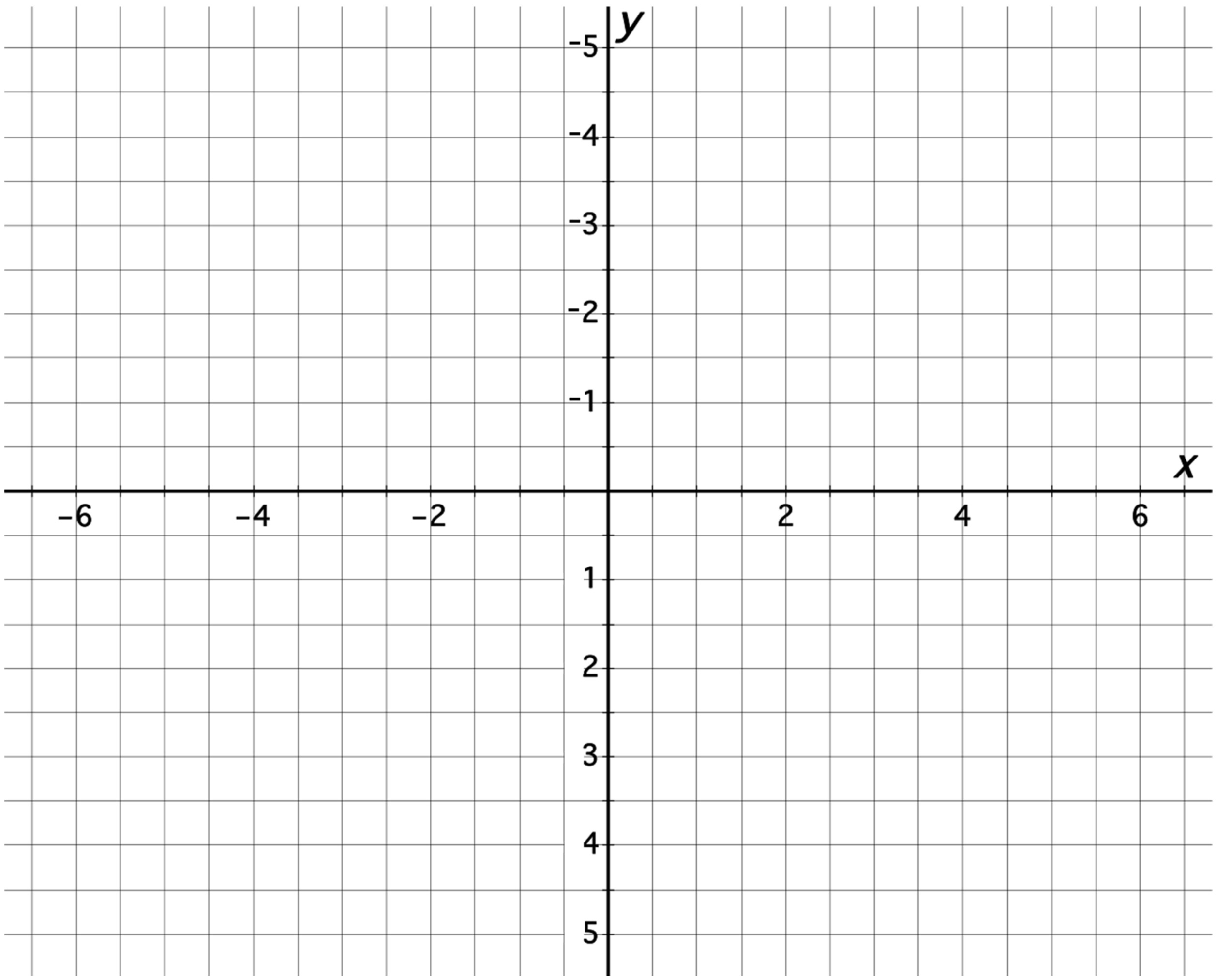
COORDINATE SYSTEM A

Sketch a graph of the function your partner describes on the axes provided.



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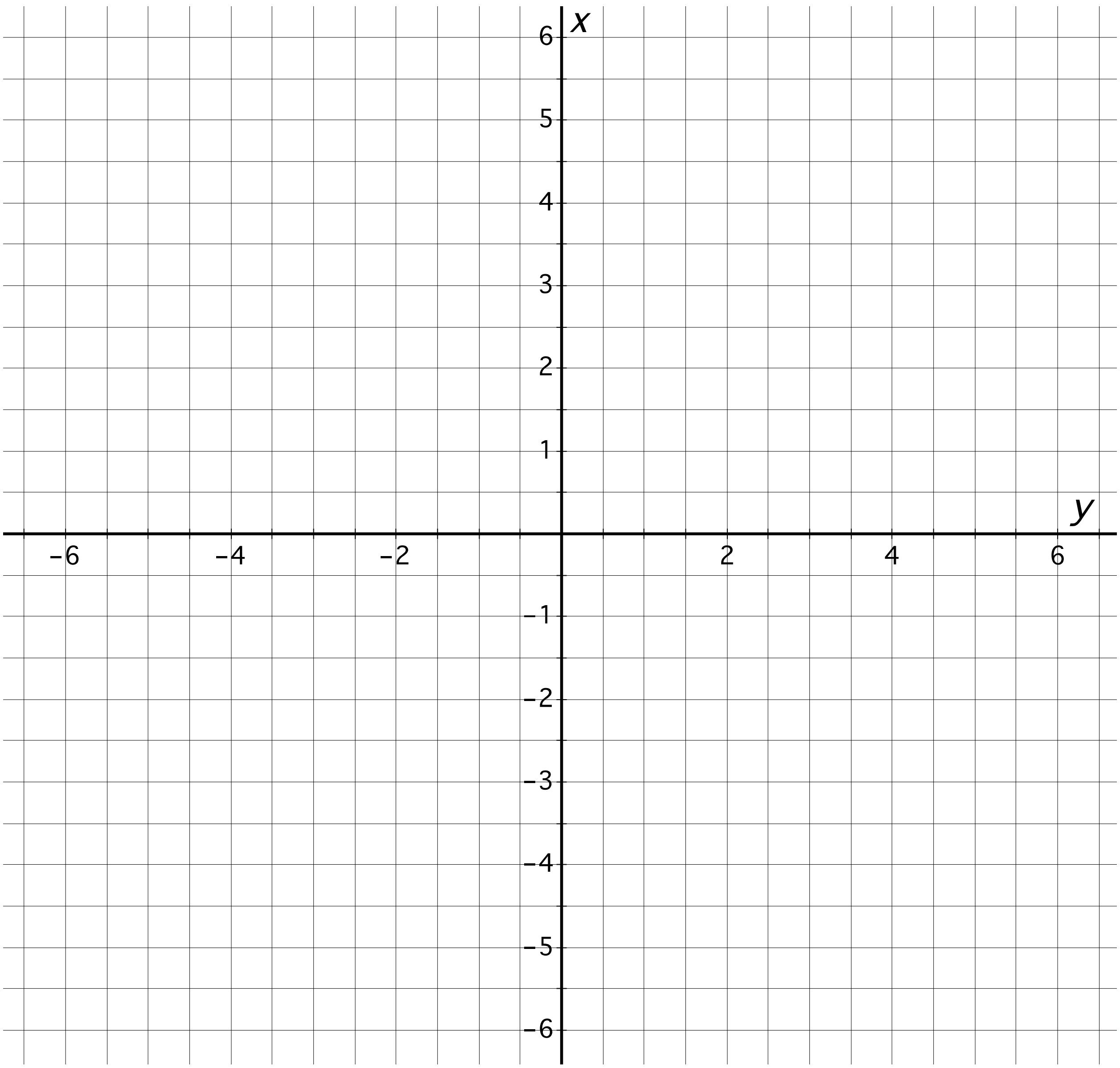
COORDINATE SYSTEM B

Sketch a graph of the function your partner describes on the axes provided.



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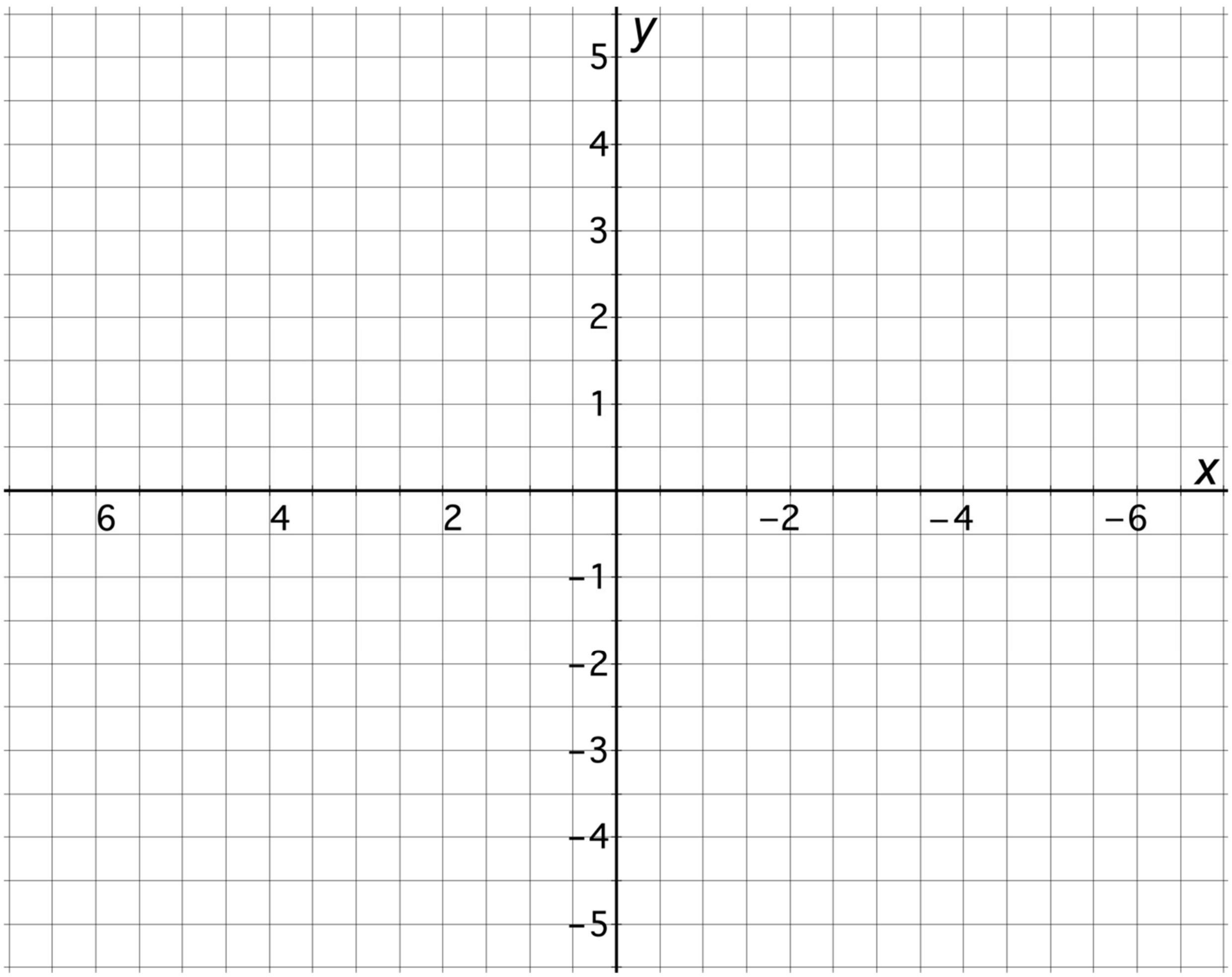
COORDINATE SYSTEM C

Sketch a graph of the function your partner describes on the axes provided.



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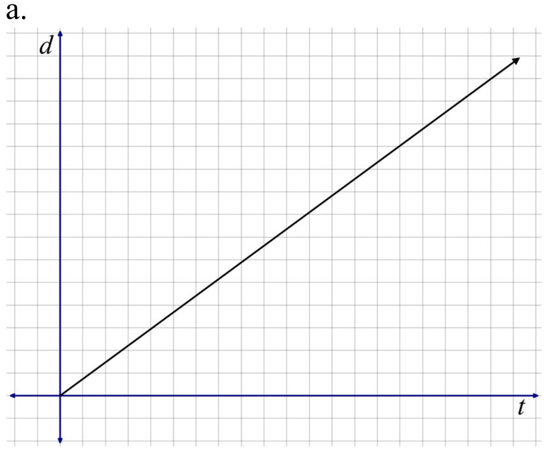
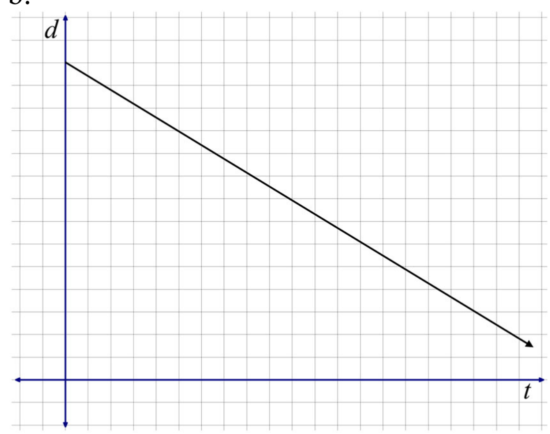
Complete the following tasks in collaboration with your partner/group.

1. A car accelerates from rest away from an intersection. Which of the following graphs most accurately represents the relationship between the car's distance d away from the intersection (in meters) and the number of seconds elapsed t since the car began accelerating away from the intersection? ***Justify your selection***.

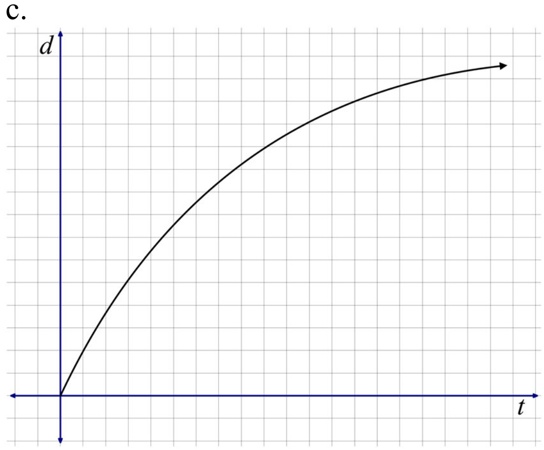
a.



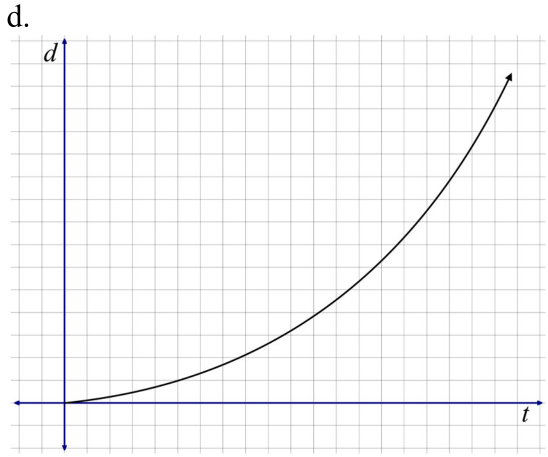
b.



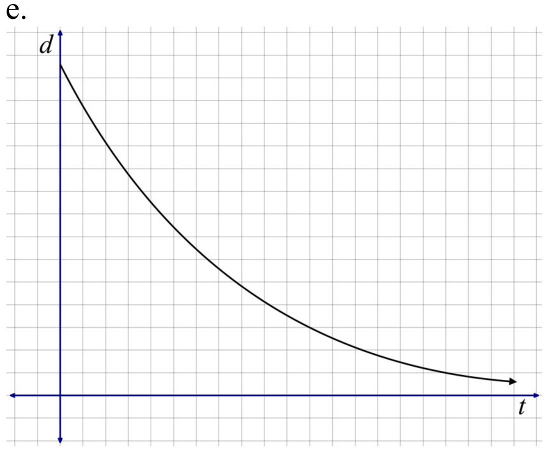
c.



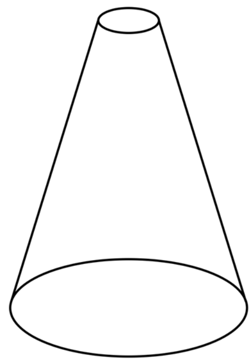
d.



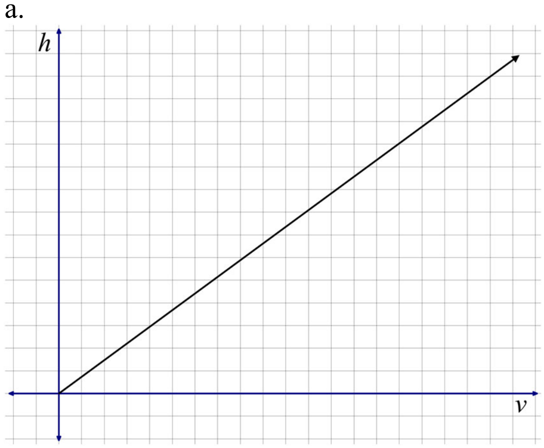
e.



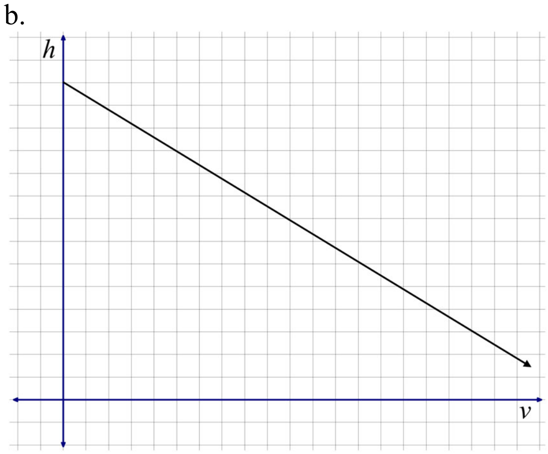
2. Imagine the bottle below being filled with water. Which of the following graphs most accurately represents the relationship between the volume v of water in the bottle and the height h of water in the bottle? ***Justify your selection***.



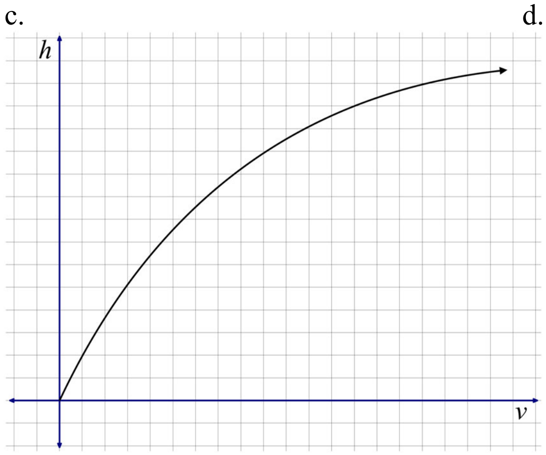
a.



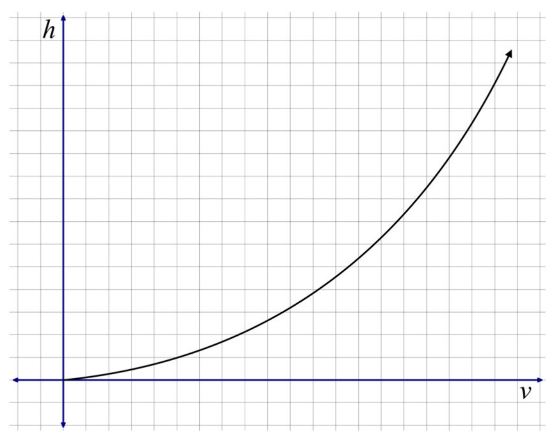
b.



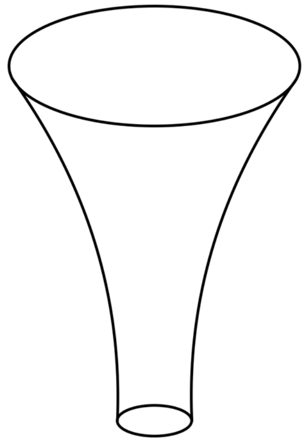
c.



d.

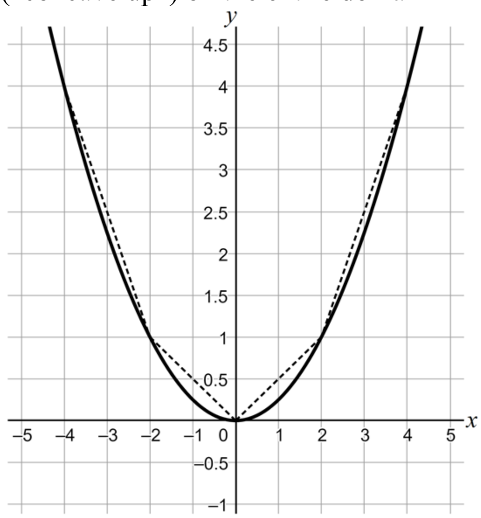


3. Imagine the bottle below being filled with water. On the axes provided, sketch a graph of the relationship between the volume v of water in the bottle and the distance d from the surface of the water to the top of the bottle.

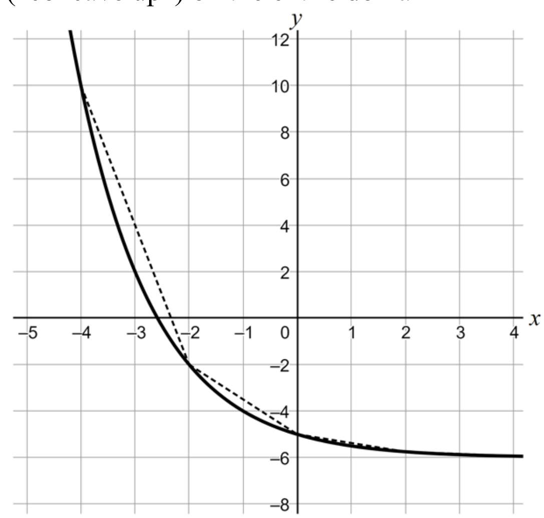


Average rates of change are measurements of how a function's output quantity changes in tandem with changes in the input quantity. ***Concavity*** is a measurement of how a function's average rate of change itself changes in tandem with changes in a function's input quantity. Examples are shown below, along with line segments whose slopes represent the functions' average rates of change over consecutive intervals.

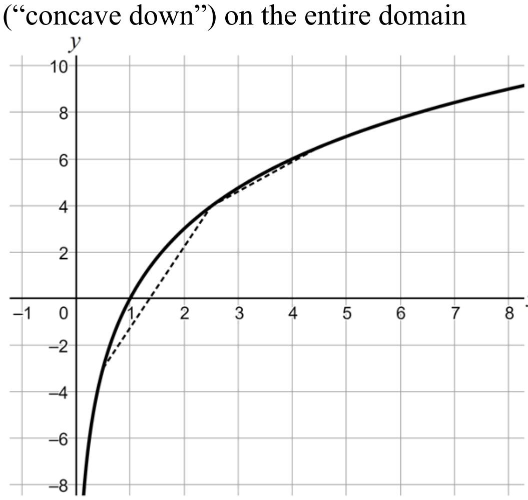
Example I: positive concavity ("concave up") on the entire domain



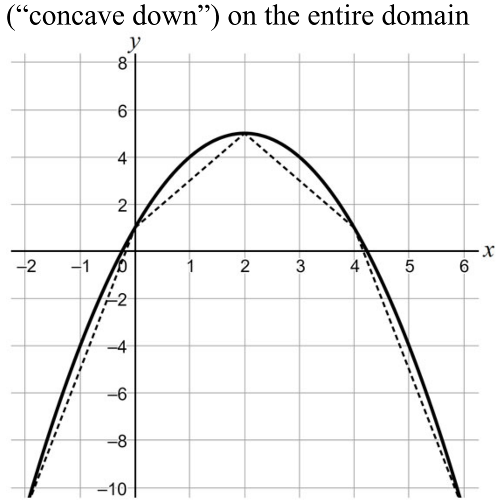
Example II: positive concavity ("concave up") on the entire domain



Example III: negative concavity ("concave down") on the entire domain



Example IV: negative concavity ("concave down") on the entire domain



Concavity

For any function f, imagine looking at an interval of the domain from $x = a$ to $x = b$ and dividing it into any number of equal-sized subintervals.

- f is said to have ***positive concavity*** on the interval (a, b) if the function's average rate over successive intervals always increases.
- f is said to have ***negative concavity*** on the interval (a, b) if the function's average rate over successive intervals always decreases.

4. a. Using the language of average rate of change, explain why the function displayed in Example I has positive concavity over the interval shown.

   b. Using the language of average rate of change, explain why the function displayed in Example II has positive concavity over the interval shown.

   c. Using the language of average rate of change, explain why the function displayed in Example III has negative concavity over the interval shown.

   d. Using the language of average rate of change, explain why the function displayed in Example IV has negative concavity over the interval shown.

4. Return to the graph that you or your partner sketched at the beginning of this activity. Over what interval(s) of the domain does the function have positive concavity? Over what interval(s) of the domain does the function have negative concavity?

5. At 5:00 pm Karen started walking from the grocery store back to her house.

a. Fill in the table by determining the number of feet Karen is from home, *d.* Then use the information in the table to determine the average rate of change of the number of feet Karen is from home with respect to time (measured in minutes since Karen started walking) on the specified intervals.

Change in the number of minutes since Karen started walking Δt	Number of minutes since Karen started walking	Number of feet Karen is from home d	Change in the number of feet Karen is from home Δd	Average rate of change of Karen's distance with respect to time
	0	118		
			–1.5	
	0.5			
			–3.2	
	1			
			–6.5	
	1.5			
			–7.1	
	2			

b. Does this function have positive concavity ("concave up"), negative concavity ("concave down"), or some combination of both on the interval $0 < t < 2$? Make sure you can justify your answer.

c. Describe how the quantities *number of minutes since Karen started walking* and the *number of feet Karen is from home* change together.

6. a. The function h is defined by $h(x) = \log_4(x)$. Use h to complete the table.

Δx	x	$h(x)$	$\Delta h(x)$	Average rate of change of $h(x)$ with respect to x
	1	0		
	2	0.5		
	4	1		
	8	1.5		
	16	2		

a. Is h an increasing or decreasing function (or a combination of the two)? Explain.

b. Does h have positive concavity ("concave up") throughout its domain, negative concavity ("concave down") throughout its domain, or a combination of the two? How do you know?

c. Is $j(x) = \log_{0.5}(x)$ an increasing or decreasing function (or a combination of the two)? Explain.

d. Does $j(x) = \log_{0.5}(x)$ have positive concavity throughout its domain ("concave up"), negative concavity ("concave down") throughout its domain, or a combination of the two? How do you know?