Activity 4: Amounts of Change – Instructor notes

The purpose of this activity is to help students reason and communicate about changes in quantities clearly in terms of context, to make rigorous arguments about how such changes are related, and to make connections between these features in the contexts and on graphs.

Active Learning: This activity requires students to construct descriptions of constant and varying rates without falling back on undefined or circular terms like "faster" or "rate." In order to do so, they must identify and articulate patterns of successive changes in two corresponding quantities (volume and height of water in various bottles). If an instructor only demonstrates these patterns while solving problems, students will often nod along to the seeming simplicity of the descriptions, but they will not develop their own ability to apply the ideas. Students need the opportunity to identify and describe the patterns of change themselves. They will struggle to get the descriptions precise and to avoid falling back on vague or intuitive terms like "steeper is faster." By holding them accountable to producing meaningful statements focused on successive amounts of change, students will develop or reinforce the MA3 reasoning described above that can later serve as a foundation for robust MA4 and MA5 reasoning.

Meaningful Applications: The affordance of this context is in the concrete imagery it provides for students to see and discuss the relationships between successive changes in two corresponding quantities. Students can concretely imagine pouring a fixed amount of water into a container of a certain size and imagine how it would spread out differently in a container of a different size. The goal is for students to then start to develop more abstract ways of conceptualizing and expressing those relationships. In this activity we focus on verbal and graphical representations of their concrete imagery about the bottles.

Academic Success Skills: Consideration and evaluation of others' ideas and refining their own mathematical argumentation are central to this activity. Rarely will students be able to identify, much less carefully articulate a rigorous explanation for the questions posed. Guide students to identify their own vague or circular language and help them to see where they only describe a relationship in one way (only about the bottle or only about the graph). Engage students in the problem-solving activity of troubleshooting and improving their own arguments.

In this activity, consider plotting height of water in a bottle vs. the volume of the water in the bottle. That is, height is on the vertical axis (dependent variable) and volume is on the horizontal axis (independent variable).

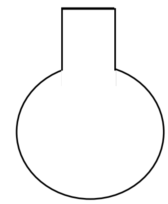
Prior to this activity, the instructor should cover the meaning of constant speed and average speed - emphasizing function notation in the process, for example as discussed in the Hoverboards & Skydiving.

We recommend assigning one of the three numbered problems to each group of students. The questions in this activity seem simple but students will struggle to express even the basic ideas of function correspondence with accurate mathematical language and notation. Hold students accountable for clearly communicating their ideas and to back them up with sound reasoning.

Make sure all students understand what others are saying, e.g., even asking students to repeat what another just said can reveal significant difficulties that should be resolved.

If you want to assign students to write up their results, provide them with the "Students' Notes" document at the end of the class session to help them understand what a complete answer should contain.

Initiating the activity:



A. Draw a graph of height vs. volume for the bottle shown to the right.

B. Is the following statement true or false? Explain how you know.

$$\text{If } V_1 < V_2 \text{ then } h(V_1) < h(V_2).$$

The first two questions may be given as a Preparation Assignment within a few days of the in-class activity and graded as formative assessment (for completion only). Students should be told to spend about 15 minutes working on these questions and we recommend collecting their responses by electronic submission prior to the remaining activity so you can quickly review the responses.

We strongly recommend allowing students to compare answers with others and discuss in small groups. Encourage them to try to listen to each other talk about any differences in their graphs then try to decide if they want to change their answer.

These questions are intended to

i) prime students' thinking about the context, quantities, notation, etc. involved in the activity so the in-class activity can begin quickly and productively,

ii) provide feedback to the instructor on any basic issues that should be resolved as the students begin work on the class activity.

Many students will just have a generally upward-trending graph. This indicates that they are considering the corresponding directions of changes in the quantities, but not amounts of change or rates.

Some students may incorporate physical features from the shape of the bottle in their graph, such as circular portions like the bottom of the bottle, or sharp corners. This will be an opportunity to help shift students toward thinking about the quantities of height and volume later in the discussions.

Some students will likely start trying to reason about corresponding amounts of change, either broken into shape-based portions of the bottle (e.g., bottom, middle, and top) or into small amounts of change in height and volume. Try to give students reasoning about small changes in heigh and volume opportunities to express their reasoning to others in the class.

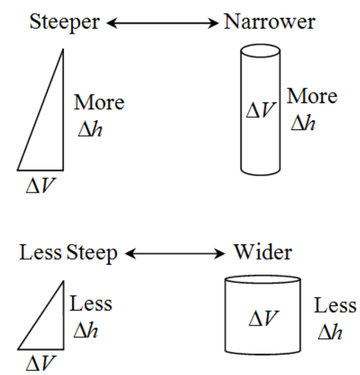
Some students may be reasoning in relatively productive ways but conflating the roles of height and volume, or switching the axes. Listen for these and try to help them be more specific to begin to make the distinction.

The mathematical characterization of an increasing function is meant to ensure students are able to interpret function notation, reason about the correspondence captured in function notation. Push students to explain the meaning of function notation in a way that captures i) quantities meaningfully and ii) the correspondence between input and output of a function.

Group Activities:

1. Steepness of the graph is related to the cross-sectional area of the bottle. Explain why a steeper graph corresponds to a narrower bottle and a less steep graph corresponds to a wider bottle. Make sure that you break down your explanation in terms of ***amounts of change*** in height and corresponding ***amounts of change*** in volume.

Students often only learn associations such as "steeper=faster" as icons, or pictoral rules. Consequently, they are unable to apply similar reasoning in other contexts or to understand the underlying mathematics. This question forces students to articulate the meaning/reason behind a typically pictoral understanding of rate of change. Students are likely to initially just repeat variations of the claim that "a narrower bottle will have a steeper graph" without realizing they are not giving a reason. A picture such as the one to the right is the key to understanding the situation and requires that they translate back and forth between meanings and relative sizes of ΔV and Δh.



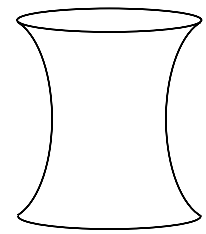
Note that the graphs are lined up to illustrate the same ΔV for both bottles, so asking students to explain how and why Δh should differ for the two bottles can help them explain the difference in the slopes of the graphs. Alternately some students may try to think about what happens for the same Δh in each bottle, which also provides a nice contrasting explanation that you can ask them to share with the class and draw the corresponding picture (showing a larger ΔV for the wider bottle).

Students will also need help seeing that a complete explanation must involve explaining why the same relationship between ΔV and Δh holds for both the bottles and the graphs. They will likely only want to articulate and justify the relationship for the bottles.

2. Describe what bottle shapes could correspond to a straight line graph. (Be creative to think of all the possibilities!) A linear graph represents a ***constant rate of change*** between the two quantities, height and volume. Explain why the bottles you described would have a straight line graph.

Asking students to reflect back on previous class discussions about the meaning of constant speed should provide traction interpreting the constant rate in this context. Note that any bottle whose horizontal cross sections have a constant area will have a constant rate of change between height and volume, but students are most likely to think about a cylinder. Push students to clearly articulate that once you fix a change in volume ΔV then all corresponding changes in height Δh will be the same as each other. Push them to articulate this relationship in terms of both bottle and the graph! Most students will initially only talk about the bottle, so they need help to see their explanation cannot relate the shapes of bottles with shapes of graphs.

3. Inflection points correspond to points where the bottle changes from getting narrower to getting wider (or vice-versa). This is because an inflection point on the graph occurs when the graph changes from getting steeper to becoming less steep (or vice-versa). Draw a graph of height vs. volume for a bottle that is narrower in the middle and explain what is happening at the inflection point using language about ***amounts of change***.



You may need to remind students that their explanation should rely on amounts of change Δh and ΔV. Students may first describe an inflection point as a place where the rate of change of height with respect to volume switches from increasing to decreasing. Reinforce this initial reasoning by noting that for the concave up portions, the graph is getting steeper indicating that the rate is increasing, but for the concave down portions, the graph is becoming less steep indicating that the rate is decreasing. But do not let the students stop there! They should then unpack the meaning of these statements in terms of amounts of change, e.g., for switching from concave up to concave down. Although students are unlikely to say something as careful or complete as the following, push them to refine their descriptions along these lines:

> "For constant size increments in volume ΔV, the changes in height Δh will be getting larger as we approach the volume where the inflection point occurs. After the inflection point, for constant size increments in volume ΔV, the changes in height Δh will be getting smaller. So the inflection point is where the changes in height switch from getting larger to getting smaller (again for constant size increments in ΔV)."