The Mathematical Inquiry Project

Request for Proposals
Collaborative Research and Development
Calculus I

The Mathematical Inquiry Project (MIP) is a statewide collaboration among mathematics faculty in Oklahoma to improve entry-level undergraduate mathematics instruction through three guiding principles:

Active Learning: Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.

For more information on the MIP Active Learning Principle, visit https://okmip.com/active-learning/

Meaningful Applications: Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.

For more information on the MIP Meaningful Applications Principle, visit https://okmip.com/applications/

Academic Success Skills: Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.

For more information on the MIP Academic Success Skills Principle, visit https://okmip.com/academic-success-skills/

Consultation

The MIP encourages those who are interested in submitting a proposal to discuss their ideas with a member of the project team, who can (1) provide feedback and advice on initial ideas, (2) connect potential CoRD members with others who are interested in working on similar ideas, and (3) offer guidance throughout the preparation and execution of the proposal. The MIP team will also organize events throughout the year to allow multiple CoRDs to present their progress and discuss ways to benefit from and integrate their approaches.

Description of CoRD products

The products produced by each Calculus I CoRD should contain four features. The MIP team is available to discuss, clarify, and provide resources for each:

1. A description of the primary conceptual goals: This analysis should include details such as the ways of understanding desired as an outcome for all students in the course, common entry points for students' understanding (including relevant supporting

concepts), a progression of challenges and solutions that students should engage through the module to develop these understandings, common pitfalls in the learning process and ways to address them, and a description of ways in which these ideas support thinking and learning throughout the entire course.

2. Instructional resources: These resources should be designed to support a coherent and productive way of reasoning about the targeted concept throughout the calculus course. The materials should include commentary for instructors about implementation of the resources and assessments that will help identify how their students have progressed relative to the targeted student learning goals. Supporting resources for instructors should also identify ways to improve their instruction relative to the targeted concept.
3. A description of how the instructional resources developed by the CoRD could be adapted to a corequisite class, including any additional resources necessary to do so.
4. Description of how the CoRD's products support the MIP components of inquiry: active learning, meaningful applications, and academic success skills.

After a successful review, the CoRD will test their products with a class or group of students and incorporate a description of the test implementation and its results, a discussion of the refinements and recommendations made based on test implementation, and short video clips with commentary to illustrate effective implementation.

Review and Revision

Once a CoRD submits a module, it will be reviewed by at least two other faculty with expertise in the topic to inform an editorial decision of "accept with minor revision," "revise and resubmit," or "reject," along with directions for revision if appropriate. (This more formal stage of review is distinct from the more informal feedback that a team might receive by interacting with an MIP team member on an ongoing basis.) After a favorable review, the CoRD will revise and pilot their module, incorporating feedback gained during the review process and submit a final module for publication on the project website.

Author Stipends

Each author in the CoRD will receive a $2,500 stipend after delivery of a complete initial draft of the module and an additional $1,000 stipend after delivery of a complete revision of the module based on the editorial decision.

Opportunities for leading regional workshops

The MIP will leverage faculty leadership and expertise developed through its Initiation Workshops and CoRDs to also develop and deliver 40 institutional and regional professional development workshops, across the state of Oklahoma. Each Regional Workshop will last a full day and engage approximately 20 mathematics faculty in implementing one or two of the modules developed by the CoRDs and ensuring familiarity with the module resources. Each workshop will be led by faculty from the respective CoRDs with support from the MIP team.

Targeted Concepts for Calculus I CoRDs

The MIP seeks to support the development of research-based instructional and curricular design principles on the following targeted concepts for Calculus I; each of these targeted concepts emerged from the work done by the faculty who participated in the Calculus I Initiation Workshop. See the ATTACHED pages for more details about each of these topics (which are listed in no particular order).

1. Functions
2. Limits
3. Local linearity, differentials, infinity, and infinitesimals
4. Rate of change
5. Continuity
6. Accumulation, integrals, and the fundamental theorem of calculus
7. Modeling

Proposal Requirements

Proposals should include each of the following:

- A cover page designating which of the targeted topics the proposed CoRD will address, names of all proposed CoRD members (2-5 people), their institutions, email addresses, and phone numbers.
- The CoRD's initial image of how it will address the four elements outlined in the "Description of CoRD Products" above.
- A description of prior experience of each CoRD member relevant to their development of the proposed module.
- A proposed timeline for the completion of the four elements outlined in the "Description of CoRD Products" above.

Proposal Length

The full text of a proposal should not exceed 2,000 words.

Proposal Submission

Completed proposals should be emailed to William (Bus) Jaco at william.jaco@okstate.edu. Proposals should be submitted by **Friday, May 27, 2022** for full consideration. The MIP will continue to accept and review proposals after this date, however we strongly encourage discussions with the project team for later submissions to avoid proposing work on topics that have already been assigned a CoRD.

The MIP plans to respond to proposals by early November. During the review of proposals, the MIP may request additional information or modifications before approval. Initial draft of modules to be reviewed will be due **January 20, 2023**.

Calculus I: Targeted Concepts for Collaborative Research and Development Teams

The information below are syntheses of the key aspects of each targeted topic developed by the faculty members who participated in the Calculus I Initiation Workshop. These syntheses are intended to serve as a guide for faculty members who are interested in or have already joined a CoRD. The MIP is available throughout the CoRD development process to answer questions and provide feedback on ideas or drafts of CoRD products.

Targeted Concept 1: Functions

Functions serve as the basic language and notation for students' experience in Calculus I. A robust understanding of functions is therefore critical for students' success in the course. Many difficulties students experience while reasoning with functions are based in a static "action view" of evaluating a function for one input at a time, typically based on an algebraic formula. In contrast, a "process view" of function in which a student can conceive of the entire process happening to all input values at once, enables them to conceptually run through a continuum of input values while attending to the resulting impact on output (e.g., see the discussion of action and process views in Oehrtman, Carlson, & Thompson, 2008). This way of thinking about the covariation of input and output values is foundational for constructing meaningful formulas and graphs when modeling relationships in applied contexts, interpreting limits conceptually or formally, and thus reasoning about all concepts defined in terms of limits (Carlson et al., 2002; Moore & Carlson, 2012; Oehrtman, Carlson, & Thompson, 2008).

Generally, modules developed by a CoRD focusing on functions should support students in at least several of the following:

- Coordinating multiple function processes (e.g., through composition, addition, or in defining an increasing function).
- Making inferences about the behavior of functions through a quantitative analysis of their symbolic representation.
- Reasoning about the behavior of functions on entire intervals in addition to single points (e.g., describing a function's behavior as input values increase continuously through the domain or finding the image of an interval)
- Reversing function processes (e.g., finding the preimage of a specified output value or interval).
- Making and comparing judgments about functions across multiple representations.
- Use dynamic graphing software (e.g., Geogebra or Desmos) to enable students to move points continuously throughout a domain and dynamically see the resulting changes in other features.
- Considering domain and range in applications.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

***Active Learning*:** Students should be engaged in tasks that go beyond treating functions as equations and provide opportunities for them to create functions to solve novel problems and invoke function notation in ways responsive to that problem-solving activity. By making students responsible for generating appropriate functions in various problem-solving situations, rather than being provided a formula to work with, they may begin to see functions as a tool for representing quantitative relationships. Additionally, students should interpret graphical representations of functions as representations of the simultaneous variation of quantities' measures in a coordinate system. This conception of graphical representations contrasts with the associations students often make between mathematical visual properties of a graph as an object (e.g., constant rate of change means "straightness"; exponential growth means "curving up"; quadratic means "U-shaped"; inverse functions mean "flip over the diagonal").

***Meaningful Applications*:** Modules may emphasize modeling and interpretation to reinforce functions as a tool to describe the world. The coordination of two quantities and univalence built into functions gain compelling meaning from natural relationships and constraints between quantities in real world situations. One may ask students to contrast the domain and range of functions based on the problem context with the domain and range derived from algebraic constraints alone. The concept of functions and function notation can be motivated and reinforced by engaging students in reasoning with and expressing quantities determined through correspondence, such as, $\Delta h = h(t + \Delta t) - h(t)$. Students should also identify and interpret key parameters in each function class in terms of the context in which it is being applied and in its various mathematical representations.

***Academic Success Skills*:** When improperly motivated, introduction of functions can seem arbitrary and unnecessarily complicated, raising a barrier to many students. Modules should help students become confident in their use of functions as a foundation of the language of mathematics and science.

Initial resources

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247-285.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education,* 33(5), 352-378.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington D.C.: Mathematical Association of America.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics, MAA Notes, Volume 73*, 27-41. Washington, DC: Mathematical Association of America.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for Research in Mathematics Education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Targeted Concept 2: Limits

Limits are often the first mathematical operation students encounter that cannot be conceived through finite computation, leaving them to negotiate spontaneous concepts of an actual infinity with infinite processes that do not end. Covariational reasoning about functional dependence is required to first conceptualize, then to coordinate, two such infinite processes quantitatively (e.g., a domain process in which $x \to a$ and a codomain process in which $f(x) \to L$, or $n \to \infty$ and $a_n \to L$). Such dynamic reasoning about functions is especially important in calculus, as the argument of a limit becomes more quantitatively complex, such as a rate of change or an accumulation.

Generally, modules developed by a CoRD focusing on limits should support students in at least several of the following:

- Operating in small neighborhoods then extending inferences beyond them.
- Treating limits conceptually in terms of approximating and refining approximations to achieve a desired level of accuracy. These ideas can be initially developed in terms of approximating instantaneous rates, such as speed, and accumulation of quantities with continuously varying rates. Subsequently, they should generalize to other contexts with the same limiting structure.
- Computer-based methods to experience and visualize the limit process.
- Repeated refinement of approximations as an experience of the limiting process

Participants of the Calculus I Initiation Workshop suggested the following ways modules on the topic of limits could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Active Learning: By actively computing several values of a difference quotient or Riemann sums for a derivative or definite integral, respectively, students can experience the limit process. Furthermore, by asking students to find such approximations to given degrees of accuracy, they must then reason in a way consistent with the formal ε-δ definition (just in a different language).

Meaningful Applications: Applications can play a particularly important role in students' experiences about limits. Specifically, when derivatives and integrals are introduced by engaging in the same problem-solving process across multiple contexts, students may recognize the common structure across all of their activity as the mathematical concept. While distance-velocity-time examples make excellent first examples, it is too often the only example students see. Such students are then likely limited to a context-dependent understanding of derivatives and integrals. Other contexts such as area and volume, mass and density, pressure and force, etc. provide good opportunities to help students generalize their understanding.

Academic Success Skills: We often see students stagnate by the fact that they think they develop enough procedural fluency to solve computational problems, yet do not understand the meanings of the limit values they are finding. Communicate to students that they need a richer understanding in a way that doesn't discourage them. Help motivate digging in to the deeper meanings of calculus rather than just the procedural "shortcuts." Encourage them to value understanding why they are working with the particular derivatives or integrals that appear in a situation.

Initial resources

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema, *Journal of Mathematical Behavior, 15*(2), 167-192.

Cornu, B. (1991). Limits. In D. Tall (Ed.). *Advanced Mathematical Thinking*, pp. 153-166. Boston: Kluwer.

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In *Making the connection: Research and teaching in undergraduate mathematics education* (Vol. 73, pp. 65-80). Mathematical Association of America Washington, D.C.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 396-426.

Szydlik, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education, 31*, 258-276.

Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Function, Limits, Infinity, and Proof. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. MacMillan Publishing Company, New York, 495-511.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*, 219-236.

Targeted Concept 3: Local linearity, differentials, infinity, and infinitesimals

Researchers (e.g., Thompson et al, 2015) have demonstrated that many calculus students' understanding of derivatives is not sufficiently grounded in robust meanings of rate of change. When prompted to explain what the derivative at a point represents, students often reply, "The slope of the tangent line." Limited to such geometric interpretations, students struggle to apply derivatives in novel contexts and to understand more advanced topics in calculus, such as linear approximation, L'Hopital's rule, implicit differentiation, related rates, Riemann sums, definite integrals, and the fundamental theorem of calculus. To address this problem, several mathematics educators (e.g., Ely, 2021) have documented the affordances of students' conceptualizing differentials as linear functions (i.e., as infinitesimal changes that vary proportionally). Related recommendations include supporting students' interpretation of "instantaneous rate of change" as "average rate of change over infinitesimally small intervals where the corresponding changes in the measures of the input and output quantities are proportional."

Generally, modules developed by a CoRD focusing on infinity and infinitesimals should support students in at least several of the following:

- Engage students with applied contexts (other than distance-velocity-time and area) that require them to reason quantitatively about derivatives and integrals.

- Use derivatives for linear approximations in instances where an instantaneous rate is known but a derivative function is unknown (as in applying Euler's method or Newton's method).
- Conceptualizing *dx* not just as a trivial part of integral notation, or as just an indicator of the independent variable for antidifferentiation, but as an infinitesimal change that when multiplied by the quantity represented by the expression in the integrand yields an incremental accumulation of some quantity.
- Understand *dy*/*dx* not just as a fraction but as the constant of proportionality that relates corresponding infinitesimal changes of covarying quantities.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Active Learning**:** Engage students in tasks that require them to leverage their understanding of the invariant multiplicative relationship between corresponding infinitesimal changes in the input and output quantities of a differentiable function to make inferences about the function, and to use information about the function to make inferences about its derivative at a point.

Meaningful Applications**:** Meaningful applications should support students' abstraction of local linearity, or a proportional relationship between differentials, conceptualized as corresponding infinitesimal changes in the input and output quantities of a differentiable function. Meaningful applications should support the ***need*** for the local linearity. It is important to remain aware that just because an application is a practical application does not automatically make it meaningful. Some criteria for meaningful applications that might support students' learning of the targeted concept include (1) contexts that require students to solve for changes (or nearby points) given a rate function they cannot simply antidifferentiate, (2) reference to quantities other than speed that are defined in terms of a variable other than elapsed time can support generalization and abstraction.

Academic Success Skills**:** Students are often overwhelmed by their perceived expectation to become proficient in applying a variety of skills and strategies required to solve different classes of disconnected problem types. Engaging students in experiences that enable them to recognize the broad applicability of conceptualizing differentials as linear functions, and the derivative as a linear map, reduces the cognitive load of memorizing an assortment of procedures for solving routine problems, and fosters students' positive affect and productive mathematical engagement.

Initial resources

Bos, H. J. M. (1974). Differentials, higher-order differentials and the derivative in the Leibnizian calculus. *Archive for History of Exact Sciences, 14,* 1–90.

Dray, T., & Manogue, C. (2003). Using differentials to bridge the vector calculus gap. *College Mathematics Journal, 34,* 283–290.

Dray, T., & Manogue, C. (2010). Putting differentials back into calculus. *College Mathematics Journal, 41,* 90–100.

Ely, R. (2010). Nonstandard student conceptions about infinitesimal and infinite numbers. *Journal for Research in Mathematics Edu- cation, 41,* 117–146.

Ely, R. (2017). Reasoning with definite integrals using infinitesimals. *Journal of Mathematical Behavior, 48,* 158–167.

Ely, R. (2021). Teaching calculus with infinitesimals and differentials? *ZDM Mathematics Education, 53*(3), 591-604.

Keisler, H. J. (2011). *Elementary calculus: an infinitesimal approach* (2nd ed.). New York: Dover Publications. **(ISBN 978-0-486-48452-5)**.

Ransom, W. R. (1951). Bringing in differentials earlier. *The American Mathematical Monthly, 58,* 336–337. https://doi. Org/10.2307/2307725.

Tall, D., (1980). Intuitive infinitesimals in the calculus. *Abstracts of short communications*, *Fourth International Congress on Mathematical Education*, Berkeley, p. C5.

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics, 48,* 199–238.

Tall, D. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM, 41*(4), 481–492.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2–3), 229–274.

Thompson, P. W., & Ashbrook, M. (2019). Calculus: Newton, Leibniz, and Robinson meet technology. Retrieved August 18, 2020, from https://patthompson.net/ThompsonCalc/.

Targeted Concept 4: Rate of change

A critical foundation for reasoning about rates of change is conceiving of changes in quantities as quantities in their own right and distinguishing such changes from the original quantities. From this foundation, students may begin to understand, distinguish, and use the meanings of constant rate of change and average rate of change in various contexts and representations. These concepts, in turn, form a foundation for students' understanding and reasoning about instantaneous rate of change in calculus. In particular, constant rate of change entails a proportional relationship between corresponding changes in the measures of the two quantities (e.g., see the conceptual analysis in Thompson, 1994). Reasoning about these changes and their proportional relationship across multiple representations can build an important foundation for further development of average and instantaneous rates. Non-quantitative interpretations of constant and average rate of change restrict students to iconic images, such as "steeper is faster." Although such pseudo-structural reasoning may be sufficient for many procedural applications of derivatives, they prevent students from productively unpacking these ideas when necessary in problem-solving situations.

Generally, modules developed by a CoRD focusing on rate of change should support students in at least several of the following:

- Help students conceive of changes in quantities as meaningful quantities in their own right (e.g., see early tasks involving describing and reasoning about changes in quantities in Carlson, Oehrtman, & Moore, 2016).
- Engage students in interpreting average rates of change as a constant rate for an auxiliary scenario with the same total changes in both quantities. These materials could reinforce and motivate the use of function notation in algebraic representations of average rates, developing the difference quotient.
- Informally introduce instantaneous rates through a context that necessitates finding average rates over progressively smaller intervals.
- Unpack rate of change statements in terms of coordinating amounts of change. Such tasks may ask students to analyze amounts of change in the function for constant increments of the independent variable (e.g., see MA3 reasoning in Carlson et al., 2002).
- Draw diagrams that represent changes in the output variable corresponding to successive increments in the input variable to help students conceptualize varying rates more robustly. Students should subsequently represent these relationships graphically and algebraically and interpret them in terms of rate of change in the problem context.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

> ***Active Learning*****:** Students in a Calculus I course will have significant experience applying procedures to solve routine problems about constant or average rate of change. Thus, it is particularly important that modules engage students in tasks that challenge these rote applications and require them to explore the underlying meanings, especially in terms of invariant multiplicative relationships between corresponding amounts of change of two quantities that vary simultaneously.

Meaningful Applications**:** Students should engage in rates of change as a natural entry point to understand, represent, and explain, how quantities change in actual situations. Correspondingly, identifying and applying key rate of change characteristics of various function types can help reinforce broader understanding of these functions and their value in appropriate modeling scenarios. Varying the contexts promotes students' development of a generalized concept of rate of change that is not bound to any single situation or representation.

Academic Success Skills**:** Exploring rate of change in-depth and in meaningful applications can help students reinforce their academic identity. Supporting students in constructing meaning for foundational mathematical ideas like rate of change allows them to develop the *expectation* that their understandings enable them to reason about novel tasks and contexts. This expectation has the potential to reduce or even eliminate the reflexive interpretations of mathematical stimuli as potential threats to one's identity, and which tend to initiate unproductive behavioral reactions (e.g., task avoidance; memorization; the unreasoned employment of coping mechanisms). In addition to supporting productive meanings for rate of change grounded in quantitative and covariational reasoning, CoRD modules could attend to reinforcing a growth mindset and encouraging perseverance by providing scaffolding that keeps students engaged without preempting their ability to develop significant solutions on their own.

Initial resources

Byerley, C. & Thompson, P. W. (2017). Secondary teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior,* 48, 168-193.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. In N. Patemen (Ed.), *Proceedings of the 2003 Meeting of the International Group for the Psychology of Mathematics Education–North America* (Vol. 2, pp. 165–172). Honolulu, HI: University of Hawaii.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*(2-3), 229-274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W., & Thompson, A. G. (1992). *Images of rate.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV.* (Vol 8, pp. 103-127). Providence, RI: American Mathematical Society.

Targeted Concept 6: Continuity

Continuity is a property of functions with several important implications. The conclusions of Rolle's theorem, the mean value theorem, the intermediate value theorem, the extreme value theorem, and the fundamental theorem of calculus all require a function to be continuous on a closed interval. Continuity is also a necessary condition for integrability and for the algebraic properties of definite integrals. It is essential that students understand the relationship between continuity and differentiability, and leverage their understanding of continuity to make strategic inferences about function behavior.

Generally, modules developed by a CoRD focusing on continuity should support students in at least several of the following:

- Connect/distinguish continuity at a point with the limit of a function at that point.
- Understanding the relationship between differentiability and continuity, namely that differentiability implies continuity.
- Relate points on non-differentiability (e.g., "corners," "cusps," "vertical tangents") to particular discontinuities of the derivative function.
- Appreciate the role of continuity as a requirement in a variety of important theorems (e.g., intermediate value theorem, mean value theorem, extreme value theorem, integrability, the fundamental theorem of calculus).
- Understand the relationship between one-sided limits, two-sided limits, the value of a function at a point, and continuity.
- Interpret various types of discontinuity (e.g., "jump," "removable," "oscillating," "infinite") in terms of limit expressions.
- Connect the limit definition of continuity to a function's graph and to intuitive notions of continuity (i.e., "you can sketch the graph without lifting your pencil").
- Understand of continuity as an extension of limits.
- Understanding the role of continuity in differentiability conditions.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

> ***Active Learning***: An instructor might engage students in active learning to support their understanding of continuity and its implications by presenting tasks that require students to find the value of a parameter to make a piecewise defined function continuous where at least one of the expressions of the function involves computing a limit using a method other than direct evaluation. Such tasks require students to apply their understanding that the limit of one expression must be equal to the value of the other at a particular input. Other tasks that might support students' active learning include asking them to sketch a graph of piecewise function given information about its one-sided limits and value at a point. Crucially, these tasks require students to coordinate the limiting value of a function with its value to assess whether the function is continuous at a particular point.

Meaningful Applications**:** Students' understanding of continuity can be reinforced and extended by engaging in applied tasks that require them to use continuity to compute the limit of a function and to interpret computational strategies for limit evaluation as an instance of leveraging the definition of continuity to evaluate a limit. Additionally, applied contexts can support students' recognition of the importance of continuity as a hypothesis in the intermediate value theorem, mean value theorem, and extreme value theorem by exploring counterexamples.

Academic Success Skills**:** Supporting students' understanding of why the conclusions of important theorems in calculus depend on a function being continuous on a closed interval enables students to interpret these conclusions as intuitive implications of continuity, not as a list of facts to be memorized and applied to solve routine exercises. Interpreting the conclusions of these theorems as intuitive implications challenges the common assumption that mathematical proficiency is based principally on one's ability to efficiently recall declarative knowledge—a perspective that increases students' uncertainty as to whether they can successfully participate in mathematics. Constructing meaning for continuity and its implications allows students to recognize that a small number of essential ways of reasoning are sufficient for engaging productively in a variety of tasks, and if students have experienced their capacity to engage in these ways of reasoning in several mathematical contexts, then they are more likely to appraise task demands as manageable, thus encouraging perseverance in problem solving and a growth mindset about mathematical ability.

Initial resources

Jayakody, G., & Zazkis, R. (2015). Continuous problem of function continuity. *For the Learning of Mathematics*, *35*(1), 8-14.

Maharajh, N., Brijlall, D., & Govender, N. (2008). Preservice mathematics students' notions of the concept definition of continuity in calculus through collaborative instructional design worksheets. *African Journal of Research in Mathematics, Science and Technology Education*, *12*(1), 93-106.

Patenaude, R. E. (2013). *The use of applets for developing understanding in mathematics: A case study using Maplets for Calculus with continuity concepts* (Doctoral dissertation, University of South Carolina).

Tall, D., & Katz, M. (2014). A cognitive analysis of Cauchy's conceptions of function, continuity, limit and infinitesimal, with implications for teaching the calculus. *Educational Studies in Mathematics*, *86*(1), 97-124.

Targeted Concept 7: Accumulation, Integrals, and the Fundamental Theorem of Calculus

Several scholars have described consequential ways of understanding the fundamental theorem of calculus and how students might apply definite integrals in modeling problems. Multiple studies (Orton, 1983; Orton, 1984; Serhan, 2015; Rasslan & Tall, 2002) suggest that students might not hold quantitative meanings for the components of an integral despite being proficient with integral calculations. Compounding this potential lack of meaning are various studies documenting the challenges students experience when attempting to apply definite integrals to contexts in physics or engineering (Sealey, 2014; Meredith & Marrongellle, 2008; Jones 2013; Jones 2015; Simmons & Oehrtman, 2017; Chhetri & Oehrtman, 2015; Bajracharya & Thompson, 2014). Other research has documented students' difficulties with coordinating the product structure $f(x_i)\Delta x$ of an accumulated quantity (e.g., Sealey, 2014). Mathematics educators have responded to these difficulties by demonstrating the effectiveness of engaging students in tasks that require them to consider how to approximate the accumulation of a quantity (or to construct a function that represents the value of an accumulated quantity) by assuming that a varying quantity (a rate, a force, etc.) is constant over some interval of its variation, and then to approximate the change in the accumulated quantity over each successive interval by computing the product of the (assumed) constant quantity and the change in the independent variable. Generally, an important instructional goal is to help students conceptualize the product of the integrand and the change in the function's independent variable as an approximation of the change of the accumulated quantity.

Generally, modules developed by a CoRD focusing on accumulation should support students in at least several of the following:

- Interpret Riemann sums quantitatively by conceiving individual terms in the sum as approximations of "bits of change" of some quantity and the sum itself as an approximate change in the value of a quantity over a particular interval of the independent quantity's variation.
- Interpret Riemann sums geometrically as approximations of bounded areas in the Cartesian plane.
- Interpret definite integrals in context as the exact change in the value of a quantity over a particular interval of the independent quantity's variation.
- Interpret definite integrals geometrically as exact values of bounded areas in the Cartesian plane.
- Leverage the idea of local linearity to approximate the accumulation of a quantity by assuming that it varies at a constant rate over small (possibly infinitesimal) intervals of the independent variable.
- Generalize their "adding up pieces" strategy of approximation in kinematic contexts to reason about quantifying accumulation in less intuitive contexts.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Active Learning: By computing successively refined approximations to the accumulation of a quantity, students may engage in a process reflecting the structure of an integral as a limit of Riemann sums. Each approximation will require students to attend to the multiplicative structure $f(x_i)\cdot\Delta x$ as an estimate of a portion of the accumulated quantity (based on an assumption of small variation in f across each subinterval). They must attend to the Riemann sum as the estimated accumulation over an interval, and refining their approximations experience the limiting process.

Meaningful Applications**:** Focusing on quantitative interpretations of all components of a Riemann sum helps students i) understand why the integral is defined as it is, ii) understand how to interpret the meaning of of an integral in context, and iii) understand how to develop an integral to model a quantity in an appropriate situation. Without a focus on quantitative reasoning, students are likely to think that an integral just adds up values of $f(x)$ or only be able to interpret integrals as "area under a curve. Furthermore, asking students to reason about a variety of contexts that involve accumulation (not just area) provides the opportunity for them to generalize their reasoning. As a result, they are better positioned to both interpret an integral in terms of its generalized, abstract mathematical structure and to create or interpret integrals in a broad range of novel contexts.

Academic Success Skills**:** Students often interpret the algebraic computations involved in applying the fundamental theorem of calculus as the "real math" that replaces ideas about limits of Riemann sums that they dismiss as the "hard way" of working with integrals. Students need experiences seeing that these underlying meanings enable powerful use of ideas about definite integrals.

Initial resources

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. In N. Patemen (Ed.), *Proceedings of the 2003 Meeting of the International Group for the Psychology of Mathematics Education–North America* (Vol. 2, pp. 165–172). Honolulu, HI: University of Hawaii.

Chhetri, K., & Oehrtman, M., (2015) The equation has particles! How calculus students construct definite integral models. *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education,* Pittsburgh, Pennsylvania.

Ferrini-Mundy, J., & Graham, K. (1994). Research in Calculus Learning: Understanding of Limits, Derivatives, and Integrals. In J. J. Kaput & E. Dubinsky (Eds.), *MAA Notes Number 33* (pp. 31–45). Mathematical Association of America.

Jones, S. (2013). Understanding the integral: Students' symbolic forms. *Journal of Mathematical Behavior, 32*, 122–141.

Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior*, *38*, 9-28.

Rasslan S, Tall D. (2002). Definitions and images for the definite integral concept. In: A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*. Norwich, UK.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*, 230-245.

Serhan, D. (2015). Students' understanding of the definite integral concept. *International Journal of Research in Education and Science*, *1*(1), 84-88.

Simmons, C. & Oehrtman, M. (2017). Beyond the product structure for definite integrals. *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA, pp. 912-919.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, *26*(2-3), 229-274.

Targeted Concept 8: Modeling

Creating and interpreting mathematical models is a critical path for students to both better understand the underlying mathematics and to be prepared to apply that mathematics in other disciplines. In calculus, students have an opportunity to reason about new types of quantities and quantitative relationships (such as instantaneous rates) and to distinguish them from previous non-calculus quantities (such as constant or average rates). Students may represent quantities and quantitative relationships as ways to mathematize a context or, conversely, to give contextual meaning to mathematical symbols. They may strategically manipulate or interpret these representations to draw inferences about a context or use the context to construct conjectures or arguments about the mathematics.

Generally, modules developed by a CoRD focusing on modeling should support students in at least several of the following:

- Help students conceive and describe real-world quantities through appropriate mathematical representations. Contexts should be chosen to make the mathematics amenable to students' intuitive reasoning that can subsequently be represented by mathematical variables, expressions, diagrams, and graphs.
- Help students conceive and describe relationships between quantities through appropriate mathematical representations. Again, contexts should be chosen to enable students to more intuitively state, justify, or question relationships between quantities, before expressing them through mathematical representations.
- Help students generalize context-specific reasoning by exploring the same underlying mathematical structure in multiple contexts, then reflecting on the similarities and differences across the resulting models (e.g., see the description of a learning trajectory across calculus leveraging abstraction across multiple contexts in Oehrtman, 2008).
- Help students abstract mathematical structure by applying concepts developed earlier tasks as tools for making sense of new situations in later tasks (e.g., see the description of levels of emergent models in Gravemeijer, Cobb, Bowers, & Whitenack, 2000).
- Develop working with quantities as a central habit of mind for students. This includes approaching any modeling situation with the initial aim to identify the relevant quantities for the given goal (e.g., see the discussion of extensive quantification in Thompson, 1994). Students should then distinguish between constant and variable quantities and identify relationships between these quantities determined by the situation. Many students will need help articulating these relationships initially using concrete numerical values for specific variable quantities, then seeing the algebra as a generalization of the multiple arithmetic expressions generated by choosing different values.
- Help students draw effective diagrams of situations with the appropriate information and level of detail to support mathematical modeling.
- Help students model changes in quantities and rates of change of one quantity with respect to another. This modeling should i) reinforce a concept of changes in quantities as meaningful quantities in their own right, ii) develop a quantitative conception of rate of change, and iii) help students identify rate of change features in contexts that correspond to particular function types to choose an appropriate algebraic form of a model (e.g., see

examples of tasks involving modeling with changes and rates of change in Carlson, Oehrtman, & Moore, 2016).

- Emphasize linear, exponential, and quadratic models that reinforce key quantitative concepts of constant rate of change, rate proportional to amount, and constant acceleration, respectively.
- Provide opportunitites for students to modeling situations with the full range of central constructs in calculus: linear relationships, limits, various types of rates, and accumulation.

Participants of the Calculus I Initiation Workshop suggested the following ways modules for this targeted topic could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Active Learning: Modules should engage students as the primary actors in creating and interpreting mathematical models. This engagement may focus on certain parts of a broader problem-solving process, but throughout the collection of CoRD resources should extend through developing, applying, and interpreting models at all stages. In doing so, they should support students in representing the products of their modeling activity using increasingly appropriate mathematical representations (terminology, symbols, expressions, graphs, procedures, etc.) and in interpreting the results of their mathematical computations in terms of the context.

Meaningful Applications: Although modeling essentially involves coordinating meanings between real-world contexts and mathematical objects and relationships, not all modeling activity productively develops conceptual understanding. In particular modules should focus students on identifying common structure across multiple modeling activities with different contexts as the source of abstracting the particular mathematical concept(s) common to them all.

Academic Success Skills: Modules should help students develop a view that mathematics is meaningful, both as a set of tools to model real-world situations, but also in the abstract, as generalizations of structures present across a wide variety of contexts. Students' engagement in this process should develop their own agency in creating these meanings and reinforce their ability to learn through persistence.

Initial resources

Council of Chief State School Officers & National Governors Association Center for Best Practices (2010). *Common core state standards for mathematics.* Common Core State Standards Initiative. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Carlson, M., Oehrtman, M., & Moore, K. (2016). *Precalculus, Pathways to Calculus: A Problem Solving Approach*, Sixth Edition. Phoenix, AZ: Rational Reasoning.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, Modeling, and Instructional Design. In Paul Cobb, Erna Yackel, & Kay McClain (Eds.) *Symbolizing and* *communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design*. Mahwah, NJ: Erlbaum and Associates. 225-273.

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior*, *45*, 95-110.

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education,* (MAA Notes, Vol. 73, pp. 65-80). Washington, DC: Mathematical Association of America.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), T*he development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.