The Mathematical Inquiry Project

Request for Proposal

Collaborative Research and Development

Functions and Modeling

The Mathematical Inquiry Project (MIP) is a statewide collaboration among mathematics faculty in Oklahoma to improve entry-level undergraduate mathematics instruction through three guiding principles:

Active Learning: Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.
For more information on the MIP Active Learning Principle, visit https://okmip.com/active-learning/

Meaningful Applications: Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.
For more information on the MIP Meaningful Applications Principle, visit https://okmip.com/applications/

Academic Success Skills: Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.
For more information on the MIP Academic Success Skills Principle, visit https://okmip.com/academic-success-skills/

Description of CoRD modules

CoRD modules should be designed to promote all three of the MIP components of inquiry: active learning, meaningful applications, and academic success skills. An overview of the module should articulate explicitly how these three components are supported.

In order to communicate the CoRD's approach to developing the targeted concepts to faculty using the MIP resources, modules should include an analysis of its primary conceptual goals. This analysis should include details such as the ways of understanding desired as an outcome for all students in the course, common entry points for students' understanding (including relevant supporting concepts), a progression of challenges and solutions that students should engage through the module to develop these understandings, common pitfalls in the learning process and ways to address them, and a description of ways in which these ideas support thinking and learning throughout the entire course.

The core of a module will be a set of instructional materials. The MIP seeks to support development of modules for entry-level college mathematics courses that develop targeted concepts as a unifying topic throughout the course. Consequently, the materials in a module will not typically consist of a sequential series of lessons, but rather provide broader instructional resources to be used throughout the course.

These materials should include assessment materials that allow an instructor both to assess how their students have progressed relative to the targeted goals and to identify ways to improve their own instruction.

As corequisite remediation for entry-level college mathematics is a critical reform in the state of Oklahoma, modules should include a description of how it would be implemented differently in a corequisite class, including any additional resources necessary to do so.
After a successful review the CoRD will pilot the module with a class or group of students and incorporate a description of test implementation and its results, a discussion of the refinements and recommendations made based on test implementation, and short video clips with commentary to illustrate effective implementation.

Review and Revision

Once a CoRD submits a module, it will be reviewed by at least two other faculty with expertise in the topic to inform an editorial decision of “Accept,” “Accept with minor revision,” “Revise and resubmit,” or “Reject,” along with directions for revision if appropriate. After a favorable review, the CoRD will revise and pilot their module, incorporating feedback gained during the review process and submit a final module for publication on the project website.

Author Stipends

Each author in the CoRD will receive a $2500 stipend after delivery of a complete initial draft of the module and an additional $1000 stipend after delivery of a complete revision of the module based on the editorial decision.

Opportunities for leading regional workshops and mentoring

The MIP will leverage faculty leadership and expertise developed through its Initiation Workshops and CoRDs to also develop and deliver 40 institutional and regional professional development workshops, across the state of Oklahoma, on teaching the new courses, incorporating applications and active learning with the modules, and addressing academic success skills. Each Regional Workshop will last a full day and engage approximately 20 mathematics faculty in implementing one or two of the modules developed by the CoRDs and ensuring familiarity with the module resources. Each workshop will be led by faculty from the respective CoRDs with support of MIP personnel who will also assist the leaders in designing the workshop activities with advice from project consultants. A goal of the Regional Workshops will be to engage all relevant faculty in hosting at nearby institutions and to develop a structure that will provide training for new faculty and continuing professional development for all faculty.

The MIP will also support 425 semester-long faculty mentoring relationships between CoRD leaders and one or two faculty who are first implementing MIP resources in a class they are teaching. A goal of these mentoring relationships is to develop institutional and regional communities whose members meet regularly and reinforce and support the cultural practices necessary for mathematics learning through inquiry.

Proposal requirements

The MIP seeks to support the development of modules on the following targeted topics for the course Functions and Modeling. See the following pages for details of each of these topics.

- Function
- Modeling and Quantitative Reasoning
- Rate of Change
- Function Classes

Proposals should include each of the following:

1. A cover page designating which of the targeted topics the proposed CoRD will address, the entry-level college course(s) for which it will develop instructional resources, names of all proposed CoRD members (3-5 people), their institutions, email addresses, and phone numbers.
2. The CoRD's initial image of how to develop the targeted concept as a unifying topic throughout the entry-level course.
3. The CoRD's initial plan to promote all three of the MIP components of inquiry: active learning, meaningful applications, and academic success skills, in their module.
4. A description of prior experience of each CoRD member relevant to their development of the proposed module.

Proposal Length

The full text of a proposal should not exceed 2,000 words.

Consultation

The MIP encourages discussions with any of the project team on the planning and preparation of a proposal. Throughout the CoRD's work, MIP project personnel will provide associated resources and advice. The MIP will also organize events throughout the year to allow multiple CoRDs to present their progress and discuss ways to benefit from and integrate their approaches.

Proposal Submission

Completed proposals should be emailed to William (Bus) Jaco at william.jaco@okstate.edu. We strongly encourage discussions with the project team to avoid proposing work on topics that have already been assigned a CoRD. The MIP generally responds to proposals within one month of their submission. During the review of proposals, the MIP may request additional information or modifications before approval.

Proposals for Functions and Modeling CoRDs may be submitted on a continuing basis until July 31, 2020 with work typically extending up to six months from the start date. Variations on topics and timing may be arranged through individual discussions with the MIP project leadership.

Functions and Modeling Targeted Topics

Function

Function is the foundational topic in Functions and Modeling. The function concept enables us to identify, analyze, and gain insight into relationships between real-world quantities that vary in tandem, and is a key prerequisite to learning subsequent ideas in this course. Accordingly, students in Functions and Modeling should develop productive understandings of function (both single- and multi-variable) that can be used flexibly amongst various real-world contexts and representations. This involves awareness and use of appropriate conventions like function notation as well as aspects of quantitative reasoning and covariational reasoning.

Participants in the MIP Initiation Workshop on Functions and Modeling suggested development of modules addressing the following areas:

1. Engage students in analyzing function relationships and concepts through multiple representations. Being able to work proficiently with each of the major function representations (e.g. formula, table, graph, words) also promotes the dynamic view that a function is much more than a way to relate specific inputs to specific outputs (i.e. instructions for how to 'convert' an input value to an output value) and reinforces the view that each representation is a different manifestation of the same relationship between quantities that are changing together (see Oehrtman, Carlson, and Thompson, 2008). Working flexibly across multiple functions representations is also valuable for understanding function *concepts* because each representation can highlight various aspects of the concept. For example, examining function composition in table and graph form might enable a student to imagine how changes in the input of one function correspond to changes in the output of the other (which students possessing only a formula-based understanding of composition would be unlikely to achieve).

2. Emphasize the concept of function as a relationship between quantities and design tasks that encourage students to reason explicitly about how a function's quantities are changing in relation to each other. Carlson et al.'s (2002) covariation framework provides details of the patterns of mental actions that support reasoning covariationally. A covariational emphasis promotes a dynamic view of function as a relationship between two changing quantities (as opposed to a static, input-output correspondence view). This emphasis also entails aspects of quantitative reasoning (which includes carefully attending to the following questions for each quantity: what is being measured, what is the measurement unit, and what does the value of the measurement?). Reasoning in this way is key for understanding the relationship between the original quantities (e.g. Moore and Carlson, 2010) and also foundational for understanding key ideas like constant and average rate of change (e.g. Thompson, 2008).

3. Have students represent the various quantities associated with a function using function notation. Note that this includes not only a proficiency with basic conventions of expressing input-output pairs in function notation, but also extends to expressions of other related quantities like change and rates of change in function notation. This representational activity can be productive because it emphasizes the common structure held by all quantities of the same type (e.g. that changes in the output quantity are all of the form $f(b)-f(a)$) and provides students with an opportunity to develop meaningful understandings of what might otherwise be rote formulas. Participants of the Initiation Workshop stressed that students should come to see function notation as an efficient and useful tool that does work for us; that is, the CoRD should design activities that enable students to see function notation as necessary for expressing mathematical ideas.

4. Leverage technology as a tool to advance students' understanding of function and related function concepts. Technology should be used to enable students to better focus on ideas and concepts, instead of only procedures and algebraic manipulation. For example, a graphing calculator (or any graphing technology) makes it easier to shift between function representations because, having entered an equation, one can view a graph or a table without getting bogged down in procedures carried out by hand (promoting the recommendation regarding the benefits of viewing functions and related concepts in multiple representations).

Participants of the MIP Workshop on Functions and Modeling suggested the following ways modules could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Meaningful Applications: Though examples of functions abound in everyday life, function is often seen by students as existing only with the confines of a mathematics class. Part of the philosophy behind Functions and Modeling is that all problems are based in real-world experiences. There are many examples of functions that students are exposed to in classes, but unless the function concept does real work in students' reasoning, they are likely to continue to confine notions of function to the classroom. Participants in the Initiation The MIP characterization of meaningful applications states that an application problem is meaningful only to the extent that it supports students in identifying mathematical relationships, justifying their reasoning, and generalizing key concepts across various contexts. Through careful instructional design, real-world applications that leverage students' real-world knowledge can become key tools for students' reasoning. For example, students can employ an analysis of a profit graph to reason about how many items yields maximal profits, break-even points, and so on.

Active Learning: Supporting students' quantitative reasoning with functions promotes insight into relationships between quantities (for example, a quantitative understanding for 'increasing' might involve the observation that the changes in output along the interval in question are all positive). Such meanings for function concepts provide rich opportunities for the MIP characterization of active learning (which includes students' selecting, performing, and evaluating actions equivalent to the concept to be learned). Tasks can pose problems about the behavior of a function's quantities in which the resolution requires attention to the desired quantitative understanding. In this way, the students have opportunities to intuitively develop function concepts as they devise their own solutions to nonroutine problems (for example, concavity can emerge in students' reasoning as they use trends they notice in the average rate of change to make predictions about the behavior of quantities).

Academic Success Skills: As function is such an integral idea upon which many future ideas depend, developing a robust, quantitative understanding of function can go a long way towards fostering students' willingness to persevere in problem solving and their identities as capable of doing mathematics. When improperly motivated, introduction of functions can seem arbitrary and unnecessarily complicated, raising a barrier to many students. Modules should help students become confident in their use of functions as a foundation of the language of mathematics and science.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, 27-42.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). PME Morelia, Mexico.

Additional Resources

Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*. (excerpt available at https://www.dropbox.com/s/ntlfnvm6ht31ycx/Sample%20Pathways%20instructor%20materials.pdf?dl=0)

Crauder, B., Evans, B., & Noell, A. (2013). *Functions and change: A modeling approach to college algebra*. Cengage Publishing.

Musgrave, S., & Thompson, P. W. (2014). Function Notation as Idiom. *Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 4, pp. 281-288). Vancouver, BC: PME. Retrieved from http://bit.ly/1p08TCG

Thompson, P. W. (2013, October). "Why use *f*(*x*) when all we really mean is *y*?". *OnCore, The Online Journal of the AAMT.*

Modeling and Quantitative Reasoning

Modeling is the process of using mathematics to describe, analyze, and gain insight into real life phenomena. It entails identifying and representing quantities and determining relationships among relevant quantities. Modeling requires careful recognition of, and attention to, the relevant quantities involved in the situation and use of either (1) patterns of covariation and/or rate of change to determine a class of functions (e.g. linear, exponential) that best model a relationship, and/or (2) prior knowledge of the relationship (e.g. physical, geometric) between these quantities to devise a model (e.g. recognizing that the volume of a box is a function of its height). Careful attention to the quantities involved is the heart of what is called *quantitative reasoning* (e.g. Thompson, 2011), and it is an indispensable component of modeling. Three key questions lie at the heart of quantitative reasoning and are instrumental in guiding the integration of quantitative reasoning into the design of instructional tasks: (1) what object is being measured?, (2) what attribute of that object are we measuring?, and (3) what is the unit of measurement?

Participants of the MIP Initiation Workshop on Functions and Modeling suggested development of modules addressing the following areas:

1. Design tasks that encourage students to develop clear (mental and physical) images of a problem scenario to identify relevant quantities and relationships among them (Moore & Carlson, 2010). We see such imagery as one piece of a larger effort to work with function relationships flexibly across the various function representations (Oehrtman, Carlson, & Thompson, 2008). For example, the change in a quantity can be represented in a diagram, as an expression in function notation (formula), the length of a line segment in a graph, an extra column in a table, and through a verbal description. Making connections between a clear and detailed mental image of the situation and other function representations supports the development of flexible quantitative understandings that are not specific to any single representation and increases the scope of situations to which a student can apply these quantitative understandings. Attending to quantities across representations also supports the development of quantitative habits of mind in which identifying and describing quantities in various forms becomes an essential way a student approaches any new problem. Such habits and skills will serve them well later in this course and in future mathematics courses.

2. Design tasks that prompt students to reason with models in various ways, including identifying relationships amongst quantities to construct their own model as well as analyzing a situation with a predetermined model. In addition to explicitly asking for measurements of specific quantities (e.g. what is the average rate of change on this interval?), which can promote overly procedural understandings if relied upon too frequently, tasks can also phrase quantities in everyday language (e.g. what is the average daily increase in your credit card balance during this time?) to emphasize that mathematics is a tool we can use to gain insight into real-world phenomena.

3. Design tasks that require quantitative reasoning. Students are adept at procedural 'shortcuts' that substantially decrease the cognitive demand of a task and hence, its pedagogical effect. Nonroutine problems (e.g. Moore & Carlson, 2012) whose solutions require attending to the 3 questions at the heart of quantitative reasoning (above) encourage more meaningful attention to patterns of covariation and rate of change (e.g. Carlson et al., 2002), which are key relationships students can leverage to develop quantitative meanings for function concepts.

4. Leverage technology for modeling, particularly involving regression. A graphing calculator can display multiple representations of a model and quickly compute regressions. The ability to efficiently generate regression equations and their corresponding graphs, scatterplots, and tables is particularly helpful in promoting the use of multiple representations. Such use of technology replaces the need for tedious calculations and procedures, which also affords additional opportunities to emphasize quantitative meanings for the various components of a regression (e.g. what do various components of a regression output measure, how are they being measured, and how do they manifest in various function representations?).

Participants of the MIP Workshop on Functions and Modeling suggested the following ways modules could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Meaningful Applications: Although modeling involves coordinating meanings between real-world contexts and mathematical representations, not all modeling activity productively develops conceptual understanding. Modules should focus students on identifying common structure across multiple modeling activities with different contexts as the source of abstracting the particular mathematical concept(s) common to them all.

Active Learning: Modules should engage students in developing, applying, and interpreting models at all stages. In doing so, they must transfer meaning both from context to mathematical representations and vice-versa.

Academic Success Skills: Modules should help students develop a view that mathematics is meaningful, both as a set of tools to model real-world situations, but also in the abstract, as generalizations of structures present across a wide variety of contexts. Students' engagement in this process should develop their agency in creating these meanings and reinforce their ability to learn through persistence.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48-59.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, *27*, 42.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming. Available at http://bit.ly/2kJv9fy.

Additional Resources

Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*. (excerpt available at https://www.dropbox.com/s/ntlfnvm6ht31ycx/Sample%20Pathways%20instructor%20materials.pdf?dl=0)

Crauder, B., Evans, B., & Noell, A. (2013). *Functions and change: A modeling approach to college algebra*. Cengage Publishing.

Rate of Change

A rate of change is a measure of how much one quantity changes with respect to another. Rates of change are an integral piece of understanding the nature of a function relationship between two quantities. Understanding rate of change provides students with tools with which they can analyze and make inferences about function behavior and hence gain insight into the real-life phenomena modeled by those functions. Rates of change that explicitly appear in Functions and Modeling include constant rate of change, average rate of change, and percentage change, but rates of change provide a means for developing quantitative understandings of function concepts as well – such as limiting value, maximum/minimum, concavity, and characterizations of function classes – suggesting that emphasizing rate of change is a primary goal throughout the entire Functions and Modeling course.

Participants of the MIP Initiation Workshop on Functions and Modeling suggested development of modules addressing the following areas:

1. Develop a quantitative understanding of rate of change (e.g. Thompson, 2011). Students should understand rate of change in a way that enables them to clearly articulate (1) what object is being measured, (2) what attribute of that object is being measured, and (3) what the unit of measurement is. Viewing changes in quantities as distinct quantities is key to developing a quantitative understanding of rate of change. Understanding function concepts in terms of amount of change is productive in itself and serves as a productive precursor to developing notions of rate of change - see Carlson et al.'s (2002) covariation framework for other examples of mental actions and imagery involving changes in quantities.

2. Establish rate of change as a conceptual tool to understand and analyze function relationships. There are several benefits to viewing rate of change as a unifying conceptual thread beyond just rates of linear functions. Focusing on quantitative meanings for function concepts can involve characterizations of the relevant properties in terms of rates of change (instead of relying solely on visual manifestations of the property – for examples, see Moore & Thompson, 2015). Students should conceive linear functions as those with a constant rate of change instead of only nonquantitative imagery such as 'those that look like a straight line.' Rate of change also allows students to develop a meaningful understanding of concavity (e.g. in terms of an increasing/decreasing average rate of change) instead of 'up like a cup, down like a

frown', or characterizing limiting behavior in terms of the average rate of change (e.g. when a function approaches a limiting value, the average rate of change tends to 0).

3. Develop the ability to reason flexibly about rates of change from each function representation, including formulas, graphs, tables, and context (see Oehrtman, Carlson, & Thompson, 2008). This is particularly useful for supplementing the perception-based understandings of function concepts (e.g. Moore & Thompson, 2015) that students hold with quantitative ones. For example, concavity might be visually evident in a graph based upon its shape, but prompting students to determine concavity for a function given in table form, or asking what attribute of a function relationship underpins the familiar "cup" shape encourages them to devise and rely on characterizations involving changes in quantities and rates of change. Recognizing and reflecting on the common structure(s) shared by amounts of change and rates of change across representations can also imbue otherwise procedural formulas and diagrams with meaning and anticipate characterizations of various function classes.

4. Use technology as a tool in advancing students' understanding of rate of change. Technology can be used to simplify the usually cumbersome task of generating function representations by hand, making it reasonable to prompt students to examine multiple representations frequently. Such use of technology can also aid visualization of rate of change. A productive conception of average rate of change involves considering the constant rate of change that would result in the same total change in output over the same input interval (see Musgrave and Carlson's (2016) conceptual analysis of average rate of change). Students may thus be asked to imagine the 'hypothetical' linear function with the aforementioned constant rate over the interval in question.

Participants of the MIP Initiation Workshop on Functions and Modeling suggested the following ways modules could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Meaningful Applications: Students should engage in rates of change as a natural entry point to understand, represent, and explain, how quantities covary in actual situations. Identifying and applying rate of change characteristics of various function types can help reinforce a broader understanding of these functions and their value in appropriate modeling scenarios. Varying the contexts promotes students' development of a generalized concept of rate of change that is not bound to any single situation or representation.

Active Learning: Students in a Functions and Modeling course may have significant experience applying rate of change in proceduralized ways. Thus, it is important that modules engage students in tasks that challenge these rote applications and require them to explore the underlying meanings, especially in terms of relationships between amounts of change. In particular, students should analyze amounts of change in ways that develops the approach as an analytic tool.

Academic Success Skills: Exploring rate of change in-depth and in meaningful applications can help students reinforce their identity as STEM students. Modules could also attend to reinforcing a growth mindset and persistence by providing scaffolding that keeps students engaged without preempting their ability to develop significant solutions on their own.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th meeting of the MAA special interest group on research in undergraduate mathematics education* (pp. 782-789). Pittsburgh, PA. Available at http://bit.ly/2kJrFK0.

Musgrave, S., & Carlson, M. (2016). Transforming graduate students' meanings for average rate of change. In *Proceedings of the 19th meeting of the MAA special interest group on research in undergraduate mathematics education*. Pittsburgh, PA. Available at http://bit.ly/2lQxrtt.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, *27*, 42.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33- 57). Laramie, WY: University of Wyoming. Available at http://bit.ly/2kJv9fy.

Additional Resources

Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*. (excerpt available at https://www.dropbox.com/s/ntlfnvm6ht31ycx/Sample%20Pathways%20instructor%20materials.pdf?dl=0)

Crauder, B., Evans, B., & Noell, A. (2013). *Functions and change: A modeling approach to college algebra*. Cengage Publishing.

Function Classes

Knowing key characteristics of the various function classes (e.g. linear, exponential, rational, polynomial) provides opportunities for students to expand their understandings of the ways in which two quantities can change together (and the associated patterns of change that might emerge). Understanding function classes in terms of these key characteristics can also expand students' understanding of functions, facilitating a shift from thinking of a function as a procedure (in which each input is 'plugged in' to a particular formula to produce an 'output') to a broader representation of an entire relationship between two quantities. This has a number of advantages, one of which is that it supports thinking about constructions involving multiple functions (e.g. function composition and combinations of functions). Another benefit of viewing a function as a unified process is that it supports students' abilities to compare and contrast the behavior of two functions against one another (in a way that is usually not possible if a student has only a computational input-output view of function). Such comparisons are integral to the Functions and Modeling course because the classification of a particular function as similar to particular class of functions provides a tool to analyze, model, and describe the behavior of a variety of real-world situations.

Participants of the MIP Initiation Workshop on Functions and Modeling suggested development of modules addressing the following areas:

1. Design tasks that emphasize quantitative and covariational characterizations of each function class. As a function is a relationship between quantities, it is propitious for students to characterize types of functions by the covariational patterns that underpin them. For example, exponential functions can be characterized covariationally in several ways, including, for fixed, uniform changes in the input quantity, (1) as functions admit a constant percentage change (alternatively, growth factor) for uniform changes in input, and (2) as functions for which the change in instantaneous rate of change (alternatively, average rate of change) is proportional to the function value. Both of these characterizations support students' ability to reason about how the quantities change together. There are several conceptual analyses in the literature that outline productive (quantitative, covariational) understandings for linear functions (e.g. Musgrave & Carlson, 2016; Thompson & Thompson, 1994; Thompson, 2008) and exponential functions (Ellis et al., 2012; O'Bryan, 2018; Thompson, 2008) that could be useful when identifying worthwhile targets of instruction.

2. Provide opportunities for students to interpret rate of change information from real-world scenarios – see Carlson et al.'s (2002) framework for examples of reasoning covariationally. Linear functions can be characterized as functions with a constant rate of change (in which the change in output is proportional to the change in input) as opposed to focusing on perceptual features (e.g. that the graph is a line). Such an image of constant rate could be used to interpolate/extrapolate unknown function values or estimate growth rates of other functions as if the function were linear. The limiting value and inflection point of a logistic function can be discussed in terms of what their respective rates of change mean within that particular situation (e.g. the inflection point occurs when the average rate of change is maximized, and the limiting value occurs when the average rate of change tends to 0, both of which underscore important information about population growth).

3. Explore key characteristics of each function class (linear, exponential, polynomial, logarithmic, and rational) through the lens of different function representations (e.g. see Oehrtman, Carlson, & Thompson, 2008). For example, reflecting on a variety of exponential functions across multiple representations can lead a student to recognize that all exponential functions can be expressed in the same algebraic form and the components of this formula correspond to important attributes of the problem situation, that the graphs of exponential functions have constant concavity (increasing/decreasing average rate of change), that the changes in the output quantity are proportional to the function value in a table, and that exponential functions can be interpreted through the lens of percentage change and/or growth factors. A student with each of these four understandings will be well-positioned to model exponential relationships, whereas a student with a minority of them will likely encounter difficulty.

4. Use technology as a tool to assist students in developing the reasoning abilities outlined above. Technology (e.g. graphing calculators) can efficiently generate additional function representations and enable students to reflect on the common features of various representations more easily than they would otherwise. Generation of these additional representations is only helpful to students if they have quantitative and covariational understandings of them. Otherwise, the features that the students reflect on and abstract across representations might be superficial and lack quantitative meaning (e.g. Thompson, 2013). Instructional designers may consider emphasizing quantitative and covariational understandings of the various function representations early in, and throughout, their modules so that students notice and attend to aspects of quantitative relationships that will enable productive real-world interpretations.

Participants of the MIP Workshop on Functions and Modeling suggested the following ways modules could address the three MIP components of mathematical inquiry (see descriptions of these components at https://okmip.com):

Meaningful Applications: Modules could emphasize modeling and interpretation to reinforce functions as a tool to describe the world. The coordination of two quantities and univalence built into the mathematical structure of functions can gain compelling meaning from natural relationships and constraints between quantities in real world situations. One may ask students to contrast the domain and range of functions based on the problem context. Students should also identify and interpret key parameters in each function class in terms of the context in which it is being applied and in its various mathematical representations.

Active Learning: Students should be engaged in tasks that go beyond treating functions as equations and that provide opportunities for them to create and interpret function models to solve novel problems. Tasks should invoke function classes and notation in ways responsive to that problem-solving activity

Academic Success Skills: When improperly motivated, introduction of functions can seem arbitrary and unnecessarily complicated, raising a barrier to many students. Modules should help students become confident in their use of functions as a foundation of the language of mathematics and science.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the jactus. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context*, *2*, 93-112.

Musgrave, S., & Carlson, M. (2016). Transforming graduate students' meanings for average rate of change. In *Proceedings of the 19th meeting of the MAA special interest group on research in undergraduate mathematics education*. Pittsburgh, PA. Available at http://bit.ly/2lQxrtt.

O'Bryan, A. E. (2018). *Exponential Growth and Online Learning Environments: Designing for and studying the development of student meanings in online courses*. Arizona State University. Available at http://bit.ly/2kl4KEK (chapter 3) and http://bit.ly/2m8Kerg (chapter 4).

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, *27*, 42.

Thompson, P. W. (2013). In the absence of meaning…. In *Vital directions for mathematics education research* (pp. 57-93). Springer, New York, NY.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education*, 279-303.

Additional Resources

Carlson, M., Oehrtman, M., & Moore, K. (2010). Precalculus: Pathways to calculus: A problem solving approach. *Rational Reasoning*. (excerpt available at https://www.dropbox.com/s/ntlfnvm6ht31ycx/Sample%20Pathways%20instructor%20materials.pdf?dl=0)

Crauder, B., Evans, B., & Noell, A. (2013). *Functions and change: A modeling approach to college algebra*. Cengage Publishing.